Mth 351 Sample Problems | Summer 2005 | (This test requires you use the provided scantron)

Bent Petersen `351u2005-sample.tex` Test date: August 10 2005 Time: 70 minutes

Instructions: $\Longrightarrow$
If you do not read the instructions, then how will you know what to do? Read them now.

- *This test is a multiple-choice test. Be sure you put your name on the scantron.*
- *You must mark your answer on the provided scantron. Fill in the appropriate bubbles on the scantron very carefully.*
- *You may use one 8.5×11 inch note sheet prepared in advance. You may write on both sides of your note sheet.*
- *Note sheets may not be shared. If you do not bring a note sheet you will have to do without any help notes.*
- *You may not use any books, notebooks, additional note sheets nor note cards.*
- *You are expected to have a simple scientific calculator or a modest graphics calculator available for use on this test. Calculators and other equipment may not be shared.*
- *You may use a simple graphics calculator but not a laptop computer nor any device capable of extensive symbolic manipulation (other than your own brain).*
- *There are 8 multiple-choice problems worth 13 points each.*

Be sure to enter all required information on the scantron.

Section Number: 001
Form Number: 001

Important Notes:

- Note that $\log(x)$ means the *natural logarithm* of x, sometimes denoted by $\ln(x)$. The logarithm with base 10 will be denoted by $\log_{10}(x)$, the logarithm with base 2 will be denoted by $\log_2(x)$, and so on.
- Return only the scantron. You may keep the test (and your note sheet).
- Make certain your calculator is set to radian mode.

The banner and instructions above are almost exactly the same as the ones you will see on test 2. In particular the problems on the test will be multiple-choice. The sample problems below, however, to save time, are not multiple choice. They do, however, have answers provided, mostly correct!

Problem 1. The function $f(x) = \sin(\cos(x))$ has a unique fixed point. Let $x_0 = 0.0$ and apply the fixed point iteration twice to compute the fixed point estimate x_2.

Ans. $x_2 = 0.61813407\ldots$

Problem 2. The interpolation polynomial $p(x)$ of degree ≤ 3 for the points $(0,2)$, $(1,2)$, $(2,4)$, and $(3,2)$ is given by $p(x) = -x^3 + 4x^2 - 3x + 2$, Let f be any four times continuously differentiable function on $[0,3]$ such that f interpolates the points $(0,2)$, $(1,2)$, $(2,4)$, and $(3,2)$. If $\left| f^{(4)}(x) \right| \leq 3$ for $0 \leq x \leq 3$ give an upper bound for $|\, f(x) - p(x) \,|$ for $0 \leq x \leq 3$.

Ans. $\frac{1}{8}\,|\, x(x-1)(x-2)(x-3) \,|$

Problem 3. Given the data points $(0,2)$, $(1,2)$, $(2,4)$, and $(3,2)$ find the least squares fit polynomial of the form $y = a(x^2+1)$.

Ans. $0.353846x^2 + 0.353846$

Problem 4. Given the data points $(0,2)$, $(1,2)$, $(2,4)$, and $(3,2)$ find the least squares fit polynomial of the form $y = ax^2 + b$.

Ans. $0.0204082x^2 + 2.42857$

Problem 5. Given the data points $(0, 2)$, $(1, 2)$, $(2, 4)$, and $(3, 2)$ find the least squares fit polynomial of the form $y = ax + b$ (so, line of regression).

Ans. $0.2x + 2.2$

Problem 6. Given the data points $(0, 2)$, $(1, 2)$, $(2, 4)$, and $(3, 2)$ find the least squares fit polynomial of the form $y = ax^2 + bx + c$.

Ans. $-0.5x^2 + 1.7x + 1.7$

Problem 7. Given the data points $(0, 2)$, $(1, 2)$, $(2, 4)$, and $(3, 2)$ find the least squares fit polynomial of the form $y = ax^2 + bx + 2$.

Ans. $-0.421053x^2 + 1.36842x + 2.0$

Problem 8. Devise an approximate quadrature rule

$$Q(f) = af(0) + bf(1/2) + cf(1) \text{ to estimate } \int_0^1 f(x)\, x^2 \, \mathrm{d}x$$

such that

$$Q(p) = \int_0^1 p(x)\, x^2 \, \mathrm{d}x$$

when $p(x)$ is a polynomial of degree ≤ 2. Then compute $Q(\cos(x))$ and the error in $Q(\cos(x))$ as an estimate of $\int_0^1 \cos(x)x^2 \, \mathrm{d}x$.

Ans. estimate= 0.2398951916, error= -0.00076156. $a = -1/60$, $b = 1/5$, $c = 3/20$

Problem 9. Devise an approximate quadrature rule

$$Q(f) = af(0) + bf(\pi/2) + cf(\pi) \text{ to estimate } \int_0^\pi f(x) \sin(x) \, \mathrm{d}x$$

such that

$$Q(p) = \int_0^\pi p(x) \sin(x) \, \mathrm{d}x$$

when $p(x)$ is a polynomial of degree ≤ 2. Then compute $Q(\mathrm{e}^x)$ and the error in $Q(\mathrm{e}^x)$ as an estimate of $\int_0^\pi \mathrm{e}^x \sin(x) \, \mathrm{d}x$.

Ans. estimate= 12.37143642, error= -0.301090102

Problem 10. For a certain function f let $T_n(f)$ be the estimate of $\int_0^2 f(x) \, \mathrm{d}x$ obtained by using the (compound) trapezoidal rule with n subintervals. Let $S_n(f)$ be the estimate of $\int_0^2 f(x) \, \mathrm{d}x$ obtained by using the (compound) Simpson's rule with n subintervals (n even in this case). If $T_6(f) = 0.308214$ and $T_{12}(f) = 0.323103$ compute $S_{12}(f)$.

Ans. 0.328066

Problem 11. If

$$B = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

then find row reduced echelon form of B.

Ans. $\text{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Problem 12. Consider a system of linear equations $A\vec{x} = \vec{b}$. After an immense number of elementary row operations we row reduce the *augmented* matrix $[A, \vec{b}]$ to the row reduced echelon form,

$$\text{rref}\Big([A, \vec{b}]\Big) = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Call the variables in the system of linear equations $x_1, x_2, \cdots$ **Part A.** Which columns are pivotal? **Part B.** Which columns are free? **Part C.** Write the general solution of the system in vector parametric form.

Ans. Columns 2,3,4,7 are pivotal. Columns 1,5,6,8 are free. The variables x_1, x_5, and x_6 may be used to parametrize the solutions. (I leave it to you to write out the solutions.)

Problem 13. A certain 3×3 matrix A has an LU decomposition with

$$L := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 1 \end{bmatrix} \quad U := \begin{bmatrix} 3 & 2 & 5 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix}.$$

Solve the system $Ax = b$ where $b = [5, 15, 10]^T$ (note ^{T} indicates the transpose).

Ans. $[9, -6, -2]^T$

Problem 14. The 5×5 Hilbert matrix A given by

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}$$

has an LU factorization $A = LU$ where

$$U = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} & \frac{1}{15} \\ 0 & 0 & \frac{1}{180} & \frac{1}{120} & \frac{1}{105} \\ 0 & 0 & 0 & \frac{1}{2800} & \frac{1}{1400} \\ 0 & 0 & 0 & 0 & \frac{1}{44100} \end{bmatrix}.$$

Compute L (recall it is normalized).

Ans.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 1 & 0 & 0 \\ \frac{1}{4} & \frac{9}{10} & \frac{3}{2} & 1 & 0 \\ \frac{1}{5} & \frac{4}{5} & \frac{12}{7} & 2 & 1 \end{bmatrix}.$$

Problem 15. If

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

find the LU decomposition of A.

Ans.

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}$$

Problem 16. Consider the linear system

$$\begin{aligned} -4x - y + 2z &= 2 \\ 2x + 5y + 2z &= 4 \\ 2x + 2y - 5z &= 3 \end{aligned}$$

Starting with an initial "guess" $[0,0,0]^T$ compute the first 5 Gauss–Jacobi iterates. Compute also the exact solution (any way you like) and use it to find the 2-norm (Euclidean norm) of the error in each iterate. Do the Gauss–Jacobi iterates appear to be converging? As a sanity check on your work, the 2-norm of the error in the third iterate appears to be about 0.06180215 . Check against your third iterate.

Problem 17. Consider the linear system

$$\begin{aligned} -4x - y + 2z &= 2 \\ 2x + 5y + 2z &= 4 \\ 2x + 2y - 5z &= 3 \end{aligned}$$

Starting with an initial "guess" $[0,0,0]^T$ compute the first 5 Gauss–Seidel iterates. Compute also the exact solution (any way you like) and use it to find the 2-norm (Euclidean norm) of the error in each iterate. Do the Gauss–Jacobi iterates appear to be converging? As a sanity check on your work, the 2-norm of the error in the third iterate appears to be about 0.03885003 . Check against your third iterate.

Problem 18. The fixed point problem

$$\begin{cases} x = f(x,y) \\ y = g(x,y) \end{cases}$$

where

$$
\begin{aligned}
f(x,y) &= \frac{x^2+3y^3+2y^2+4y+1}{15} \\
g(x,y) &= \frac{3x-2x^2+4y-2y^2+3}{15}
\end{aligned}
$$

has the solution $(\overline{x},\overline{y}) = (0.1645436899, 0.2966770274)$. (There is another solution, $(-2.337069, -2.441771)$, but it does not concern us here.) Use the initial point $(x_0, y_0) = (0, 0)$ and compute 8 fixed point iterates (x_n, y_n) by the method of simultaneous displacements (Jacobi iteration). At each step compute also the error

$$
\left((\overline{x}-x_n)^2 + (\overline{y}-y_n)^2\right)^{1/2}.
$$

Do you appear to have convergence?

Problem 19. Repeat the previous problem but this time use the method of successive displacements (Seidel iteration). Compare the errors to the ones observed in the previous problem.

Problem 20. Let A be an invertible 3×3 matrix and suppose

$$
A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}
$$

has the solution

$$
\begin{aligned}
x &= 10a+5b-8c \\
y &= -9a-4b+7c \\
z &= 3a+b-2c
\end{aligned}
$$

. Find A.

Ans.

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 3 & 5 & 5 \end{bmatrix}.
$$

The fi nal test will have 8 to 10 problems on it. The problems will be multiple choice and need not resemble any of the problems above (though some probably will).