Problem 18.3

Use NEWTON's method for systems to approximate a solution to the system of equations

$$\begin{aligned} x^2 + y^2 + z^3 &= 3 \\ x + 2y - 2z &= 0 \\ x^3 + y^3 - 3z^2 &= 2. \end{aligned}$$

Use the initial point $[1.0, -1.0, 0]$ and iterate three times. Given that the actual solution is $[1.44825, -0.95663, -0.23250]$ compute the error in each of your iterates. Comment.

Solution to problem 18.3

The residual vector at the point $v = [x, y, z]$ is

$$R(v) = [x^2 + y^2 + z^3 - 3, x + 2y - 2z, x^3 + y^3 - 3z^2 - 2]$$

The Jacobian at the point $v = [x, y, z]$ is

$$J(v) = \begin{bmatrix} 2x & 2y & 2z \\ 1 & 2 & -2 \\ 3x^2 & 3y^2 & -6x \end{bmatrix}.$$

Newton's method is the iteration

$$\begin{aligned} J(v_n)\Delta v_n &= -R(v_n) \\ v_{n+1} &= v_n + \Delta v_n \end{aligned}$$

and we are given

$$v_0 = [1.0, -1.0, 0].$$

In the problem we are also give the true solution (not normally available)

$$v_{\text{true}} = [1.44825, -0.95663, -0.23250].$$

The error in $v_n$ is then given by

$$\text{err}_n = \| v_{\text{true}} - v_n \|_2 .$$

The residual at $v_n$ is given by

$$\text{res}_n = \| R(v_n) \|_2 .$$

Now

$$J(v_0) = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 2 & -2 \\ 3 & 3 & 0 \end{bmatrix}.$$

and $R(v_0) = [-1, -1, -2]$. We solve the linear system

$$J(v_0)\Delta v_0 = -R(v_0)$$

and obtain

$$\Delta v_0 = [.5833333334, .8333333325e-1, -.1250000001].$$

Thus our first iterate is

$$v_1 = v_0 + \Delta v_0 = [1.583333333, -.9166666668, -.1250000001].$$

For the error in $v_1$ and the residue at $v_1$ we compute easily

$$\text{err}_1 = 0.1772027788 \qquad \text{and} \qquad \text{res}_1 = 1.207982121$$

For the next step

$$J(v_1) = \begin{bmatrix} 3.166666666 & -1.833333334 & -.2500000002 \\ 1 & 2 & -2 \\ 7.520833329 & 2.520833334 & .7500000006 \end{bmatrix}.$$

and $R(v_1) = [.3628472210, -.8e-9, 1.152199071]$. We solve the linear system

$$J(v_1)\Delta v_1 = -R(v_1)$$

and obtain

$$\Delta v_1 = [-.1358878728, -.2422962813e-1, -.9217356494e-1].$$

Thus our second iterate is

$$v_2 = v_1 + \Delta v_1 = [1.447445460, -.9408962949, -.2171735650].$$

For the error in $v_2$ and the residue at $v_2$ we compute easily

$$\text{err}_2 = .02197945337 \qquad \text{and} \qquad \text{res}_2 = 0.580853723$$

If we iterate again we obtain

$$v_3 = [1.438310885, -.9395627198, -.2204072772]$$

and

$$\text{err}_3 = .02315841111 \qquad \text{and} \qquad \text{res}_3 = .0003251737$$

The residual looks great but the error got larger! Why? Well, it turns out there is an error in the statement of the problem. The true solution should have been $[1.438268047, -.9395705255, -.2204365018]$. **Exercise:** Recompute the errors using the correct true solution. Then do a couple more iterates to check that you understand the process.

Note in a real problem the true solution is not known. After all, we are trying to find the solution. All we have to check our progress is the residuals and the increments $\|\Delta v_n\|_2$.