cells	objects	specification	number	notes
n, dist	m, dist	Place all objects in the cells. Some cells may be empty.	n^m	For each object choose a cell in n ways to put it in.
n, dist	m types	Place exactly one object in each cell. Assume we have an unlimited number of each type.	m^n	For each cell we have m choices of type.
n, dist	m, dist	m_k objects in k^{th} cell, $k = 1 \cdots, n$. Here $m = m_1 + \cdots + m_n$.	$\frac{m!}{m_1! \cdots m_n!}$	Place the objects in order, then permute the contents of each cell.
n, indist	m, dist, $m = m_1 + \cdots + m_n$	m_k objects in k^{th} cell. Let p_i, $i = 1, \ldots, h$ be the number of occurences of each distinct value of the m_k.	$\frac{m!}{m_1! \cdots m_n!} \frac{1}{p_1! \cdots p_h!}$	Permute the cells having the same cardinality since we can't distinguish them.
n, dist, $n \geq m$	m, $m = m_1 + \cdots + m_h$	m_k objects of type k. Place all objects, at most one per cell.	$\frac{m!}{m_1! \cdots m_h!} \binom{n}{m}$	Choose successively the cells to receive an object of type k, including $n - m$ objects of "virtual" type.
n, dist	n, dist	Place one object in each cell.	$n!$	Think of the cells as being in order.
n, dist	m, dist, $m \leq n$	Place at most one object in each cell.	$\frac{n!}{(n-m)!}$	Introduce $n - m$ virtual distinguishable object in $(n - m)!$ ways in the $n - k$ empty cells.
n, indist	m, dist, $m \geq n$	Place at least one object in each cell.	$\mathrm{S}(m, n)$	The cells partition the set of m objects into n (nonempty) subsets - so Stirling number of second kind.
n, dist	m, dist, $m \geq n$	Place at least one object in each cell.	$n!\,\mathrm{S}(m, n)$	Permute the n cells. This is also the number epimorphims of a set with m elements onto a set with n elements.
n, indist	m, dist, $m \geq n$	Place all the objects in the cells. Some cells may be empty.	$\sum_{k=1}^{n} \mathrm{S}(m, k)$	The cells partition the set of m objects into n or fewer (nonempty) subsets.
n, indist	n, dist	Place all the objects in the cells. Some cells may be empty.	$\mathrm{Bell}(n) = \sum_{k=1}^{n} \mathrm{S}(n, k)$	The cells partition the set of n objects into n or fewer (nonempty) subsets. So the Bell number.
n, dist	m, indist	Place all the objects in the cells. Some cells may be empty.	$\binom{m+n-1}{n-1}$	Choose the $n - 1$ positions from the $m+n-1$ to locate walls separating the objects.
n, dist	m, indist, $m \geq n$	Place all the objects in the cells and at least one in each cell.	$\binom{m-1}{n-1}$	Put one object in each cell and use the previous result with $m - n$ objects.
n, indist	m, indist, $m \geq n$	Place all the objects in the cells and at least one in each cell.	N(m,n)	The number of partitions of m containing n terms.
n, indist	m, indist, $m \geq n$	Place all the objects in the cells.	$\sum_{k=1}^{n} \mathrm{N}(m, k)$	The number of partitions of m containing at most n terms.

Note: "Dist" means distinguishable and "indist" means indistinguishable.

The number of sequences $j_1, j_2, \cdots, j_k$ of integers with

$$1 \le j_1 < j_2 < \cdots < j_k \le n$$

is the same as the number of ways to choose k distinct numbers from $\{1, 2, \cdots, n\}$. But this number is

$$\binom{n}{k}.$$

The number of sequences $j_1, j_2, \cdots, j_k$ of integers with

$$0 \le j_1 < j_2 < \cdots < j_k \le n$$

is the same as the number of ways to choose k distinct numbers from $\{0, 1, 2, \cdots, n\}$. But this number is

$$\binom{n+1}{k}.$$

The number of sequences $j_1, j_2, \cdots, j_k$ of integers with

$$1 \le j_1 \le j_2 \le \cdots \le j_k \le n$$

is the same as the number of sequences $j_1, j_2, \cdots, j_k$ of integers with

$$1 \le j_1 < j_2 + 1 < j_3 + 2 < \cdots < j_k + k - 1 \le n + k - 1.$$

But this number is

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

This beautiful argument is due to Euler.

The number of sequences $j_1, j_2, \cdots, j_k$ of integers with

$$0 \le j_1 \le j_2 \le \cdots \le j_k \le n$$

is

$$\binom{n+k}{k} = \binom{n+k}{n}$$

by the previous example with n replaced by $n + 1$.

Consider the equation (for nonnegative integers)

$$x_1 + x_2 + \cdots x_n = k.$$

If we think of $x_1, \cdots, x_n$ as distinguishable cells into which we want to place k indistinguishable objects (or ones) we see that the equation has

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

nonnegative integer solutions.

Consider the inequality

$$x_1 + x_2 + \cdots x_n \le m.$$

This inequality is equivalent to the equation

$$x_1 + x_2 + \cdots x_n + y = m.$$

On replacing n by $n+1$ above we see the number of solutions is

$$\binom{n+m}{m} = \binom{n+m}{n}.$$

The last two results imply

$$\sum_{k=0}^{m} \binom{n+k-1}{k} = \binom{n+m}{m}.$$

If we multiply out $(x_1 + x_2 + \cdots x_n)^m$ the number of times that a given multinomial occurs is the same as the number of arrangements of m objects in m distinguishable cells, one per cell, where there are k_j objects of type j (the x_j), and we distinguish the objects only by type. Thus (by the fifth item in our table) we have the multinomial theorem

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{k_1+\cdots+k_n=m} \frac{m!}{k_1!k_2!\cdots k_n!}\, x_1^{k_1} \cdots x_n^{k_n}.$$

The number of terms in this sum is the number of nonnegative integer solutions of the equation $k_1+\cdots+k_n = m$, that is

$$\binom{m+n-1}{n-1} = \binom{m+n-1}{m}.$$

Note setting all the x_j to 1 we obtain

$$n^m = \sum_{k_1+\cdots+k_n=m} \frac{m!}{k_1!k_2!\cdots k_n!}.$$

We can also see this result combinatorially. After all, the sum is just the total number of ways to place m objects, of n types in m distinguishable cells (since we sum over all possibilities), exactly one object per cell. Thus n^m by item two in our table (m and n interchanged).

Corrections and comments are welcome.

Bent E. Petersen	phone numbers
Department of Mathematics	office (541) 737-5163
Oregon State University	
Corvallis, OR 97331-4605	fax (541) 737-0517

petersen@math.orst.edu
bent@alum.mit.edu
http://www.onid.orst.edu/~peterseb
http://www.peak.org/~petersen