Mth 355 (399) | Take-Home Exam | Due: Thursday March 20, 2003 3:30 PM or earlier

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You may use any books, notes, calculators, computers - whatnot. Your solutions should be submitted by email, with or without attachments as appropriate, and should include at least a sketch of your work. Part of the test is to figure out how to communicate your solutions. You may find a Maple Worksheet attachment convenient.

The last 2 problems are Maple problems. The solutions must be submitted on a Maple worksheet (as an email attachment).

Your email must include the statement "I certify that I have not received any help from anyone, other than possibly from Bent Petersen, for any of the work submitted for this exam. Moreover, I certify that I have not provided any help to anyone else, nor engaged in any discussion of the content of the exam."

Problem 1. *(20 points).* Find the number of sequences j_1, j_2, j_3, j_4 with

$$1 \le j_1 \le j_2 < j_3 \le j_4 \le 128.$$

(Note the inequalities carefully.)

Problem 2. *(20 points).* For a smooth real-valued function f of n real variables the theorem on the equality of mixed partial derivatives states that the partial derivatives do not depend on the order in which we differentiate, but just on the number of times we differentiate relative to each variables. There are no other restrictions on the values of the partial derivatives. Given these facts, how many distinct partial derivatives of order $\le m$ are generally possible for a smooth function of n real variables?

Problem 3. *(20 points).* How many integers between 37 and 20,349 are divisible by 4 and by 6, but not by 5?

Problem 4. *(20 points).* Let d_n be the number of derangements of n distinct objects. Find the generating function for the sequence

$$\frac{d_n}{n!}.$$

Problem 5. *(20 points).* An urn contains 6 red marbles, 4 green marbles, 7 white marbles and 5 blue marbles. In how many ways can we select a set 12 marbles containing an odd number of white marbles?

Problem 6. *(20 points).* Solve the recurrence relation

$$a(n) = 2a(n-1) + a(n-2) - 2a(n-3), \quad a(0) = 1, a(1) = 3, a(2) = -1.$$

Problem 7. *(20 points).* Solve the recurrence relation

$$a(n) = 2a(n-1) - 4a(n-2) + 8a(n-3), \quad a(0) = 4, a(1) = 8, a(2) = 0.$$

For what values of n is $a(n) = 0$?

Problem 8. *(20 points).* We may think of the n-cube as being constructed from two copies of the $(n-1)$-cube by joining the 2^{n-1} vertices of the one copy with the corresponding vertices of the other copy by new edges. Find a recurrence relation for the number $a(n)$ of edges of the n-cube. Noting $a(1) = 1$ solve for $a(n)$.

Problem 9. *(20 points).* **[Maple Problem]** Write a Maple procedure funm(f,A) which takes as arguments a function f and a diagonizable square matrix A of any size and returns $f(A)$ defined as follows: If $SAS^{-1} = D$ is diagonal, say $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, then $f(D) = \mathrm{diag}(f(\lambda_1), \cdots, f(\lambda_n))$ and $f(A)$ is defined to be $S^{-1}f(D)S$. Use D:=jordan(A,'S'); from the linalg package to find D and S.

Verify your work by comparing A^3 and funm(x–> x^3,A) for some symmetric matrix A. Then compute the absolute value of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix},$$

that is, compute funm(abs,A).

Problem 10. *(20 points).* **[Maple Problem]** Write a Maple procedure pts() which given any number of lists $L_1, \cdots, L_n$ of arbitrary lengths returns a list L such that the k^{th} entry of L is $[L_1[k], L_2[k], \cdots, L_n[k]]$, where $k = 1, 2, \cdots, N$ where N is the minimum length of the L_k.

Note pts($[1, 2, 3], [4, 5, 6], [7, 8, 9], [10, 11, 12]$); should return

[[1, 4, 7, 10], [2, 5, 8, 11], [3, 6, 9, 12]]

and pts($[1, 2, 3], [4, 5, 6], [7, 8], [10, 11, 12]$); should return

[[1, 4, 7, 10], [2, 5, 8, 11]].

If $L_1, \cdots, L_n$ are lists of the same length what is the result of pts(op(pts($L_1, \cdots, L_n$)))?
