

Complex Variable – Mth 417

Archive – Spring 1998 Files

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This archive contains the sample problems, assignments and final exam from Mth 417 Spring 1998. The original test and assignment instructions, headers and formatting have not been preserved.

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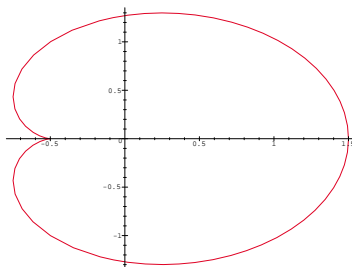
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1 Assignment 1

Problem 1. Let

$$f(z) = z + \frac{1}{2}z^2.$$

Show that f maps the unit disk $D(0,1) = \{z \mid |z| < 1\}$ one-to-one onto the interior of a “cardioid” (see illustration below – it may not coincide with the definition you saw in calculus). Show the mapping is not one-to-one in $D(0,r) = \{z \mid |z| < r\}$ if $r > 1$.



2 Assignment 2

Problem 2. Let f be an entire function. Let $C > 0$ and n be real constants. Suppose

$$|f(z)| \leq C |z|^n$$

for all $|z| \geq 1$. Use the Cauchy inequalities to deduce that f is a polynomial of degree at most n .

Problem 3. Let f be an entire function. Let A and B be real constants. Suppose

$$\Re f(z) \leq A + B \log(1 + |z|)$$

for each z . Show that f must be constant. (Hint: If you get a good enough start, you may find the fundamental theorem of algebra useful to finish.)

Problem 4. Let $a > 1$ be a real number. Give a careful proof that

$$\int_0^{2\pi} \frac{d\theta}{a + \sin \theta} = \frac{2\pi}{\sqrt{a^2 - 1}},$$

where the radical indicates the positive square root. Your proof should make use of the last assertion in the previous sentence.

3 Sample Problem Set 1

Problem 5. Find the Laurent series of $\frac{1}{z^4 - 1}$:

A: In the annulus $|z| > 1$.

B: In the annulus $0 < |z - i| < \sqrt{2}$.

C: In the annulus $\sqrt{2} < |z - i| < 2$.

D: In the annulus $2 < |z - i|$.

Problem 6. Show if $|z| > 1$ then $\arg\left(\frac{z+1}{z}\right) \neq \pi, \text{ mod } 2\pi$. Conclude if \log is the principal branch of the logarithm, then $\log\left(\frac{z+1}{z}\right)$ is defined and analytic in the annulus $|z| > 1$. Find its Laurent series in this annulus.

Problem 7. Find the residue of $\frac{\sin z}{z^6}$ at the origin. Compute $\int_{\gamma} \frac{\sin z}{z^6} dz$, where $\gamma(t) = e^{it}$, $0 \leq t \leq 4\pi$.

Problem 8. Compute the residue of $e^{1/z} \sin z$ at the origin. Compute $\int_{\gamma} e^{1/z} \sin z dz$, where $\gamma(t) = 2e^{it}$, $0 \leq t \leq 2\pi$.

Problem 9. Find the residue of

$$\frac{z^3}{(z+1)^4(z-2)^2}$$

at the point $z = -1$.

Problem 10. Find the residue of

$$g(z) = \frac{z^2 + 4z + 2}{z^3 - z^2 - 2z}$$

at each isolated singularity. If γ is a closed contour in \mathbb{C} and γ does not meet the singularities of g then what are the possible values of $\int_{\gamma} g(z) dz$?

Problem 11. (Tricky standard example!) Let $0 < \alpha < 1$. Use the residue theorem to evaluate

$$\int_{\gamma} \frac{e^{\alpha z}}{1 + e^z} dz$$

where γ is the boundary of the rectangle $|x| < N$, $0 < y < 2\pi$. Deduce

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x} dx = \frac{\pi}{\sin(\alpha\pi)}.$$

Problem 12. Use the residue theorem and a limit argument to compute

$$\int_{-\infty}^{\infty} \frac{x^4}{1 + x^6} dx.$$

Problem 13. Let $\gamma_r(t) = re^{it}$, $0 \leq t \leq 2\pi$, $r > 0$. Use the residue theorem to evaluate

$$\int_{\gamma_r} \frac{dz}{(z+1)(z+2)(z+3)}$$

as a function of r , $r \neq 1, 2, 3$.

Problem 14. Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} 4^n z^n.$$

Problem 15. Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} n! n^{-n} z^{2n}.$$

Problem 16. Show the function g defined by

$$g(z) = \frac{z}{e^z - 1}$$

has a removable singularity at the origin and can therefore be expanded in a power series about the origin. Explain how you know that this power series has radius of convergence 2π . The Bernoulli numbers B_n are defined by $B_n = g^{(n)}(0)$ so

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n!} B_n z^n, \quad |z| < 2\pi.$$

Show that $g(z) + \frac{1}{z}$ is an even function. Conclude that $B_1 = -\frac{1}{2}$ and $B_n = 0$ if $n \geq 3$ is odd. Now show that

$$g(z) - g(2z) = \frac{z}{e^z + 1}$$

and conclude

$$\frac{z}{e^z + 1} = \sum_{n=1}^{\infty} \frac{1 - 2^n}{n!} B_n z^n = \frac{z}{2} + \sum_{n=1}^{\infty} \frac{1 - 2^{2n}}{(2n)!} B_{2n} z^{2n}.$$

where the series has radius of convergence π . Replace z by $2z$ to deduce

$$z \frac{e^{-z} - e^z}{e^{-z} + e^z} = \sum_{n=1}^{\infty} \frac{(1 - 2^{2n})2^{2n}}{(2n)!} B_{2n} z^{2n}$$

where the series has radius of convergence $\frac{\pi}{2}$. Conclude

$$\begin{aligned} \tan z &= \sum_{n=1}^{\infty} \frac{(1 - 2^{2n})2^{2n}}{(2n)!} B_{2n} (-1)^n z^{2n-1} \\ &= z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \frac{17}{315} z^7 + \frac{62}{2835} z^9 + \frac{1382}{155925} z^{11} \\ &\quad + \frac{21844}{6081075} z^{13} + \dots, \quad \text{for } |z| < \frac{\pi}{2}. \end{aligned}$$

The Bernoulli numbers can be computed, and many their properties can be deduced, from properties of the generating function g . For example $e^z g(z) = z + g(z)$ implies

$$B_n = \sum_{k=0}^n \binom{n}{k} B_k.$$

This expression implies $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$ and $B_{14} = \frac{7}{6}$. In this manner we obtain the first few terms in the expansion of $\tan z$ as above. The simple expressions for the first few Bernoulli numbers are a bit misleading. For example, we have

$$B_{80} = \frac{-4603784299479457646935574969019046849794257872751288919656867}{230010}.$$

Problem 17. Let f be an entire function. Suppose there is a constant M such that

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right| d\theta \leq M, \quad \text{for each } r > 0.$$

Prove that f is constant. (Hint: Cauchy integral formula.)

4 Sample Problem Set 2

Problem 18. Show if $\Re a < 0$ and $\Re z \geq 0$ then

$$\Re \left(\frac{1}{z-a} \right) > 0.$$

Conclude if $f(z)$ is a polynomial and all the roots of $f(z)$ have negative real parts then all the roots of the derivative $f'(z)$ have negative real parts.

Hint: Show $\Re \left(\frac{f'(z)}{f(z)} \right) > 0$ if $\Re z \geq 0$.

Problem 19. Let $\alpha \neq 0$ and β be complex numbers. Show that

$$L = \{ z \in \mathbb{C} \mid \Re(\bar{\alpha}z + \beta) = 0 \}$$

is the line with normal α and passing through the point $-\frac{\alpha\beta}{|\alpha|^2}$.

Problem 20. Let $\alpha \neq 0$ and β be complex numbers. Let $f(z)$ be a polynomial with all of its roots in the half-plane

$$H = \{ z \in \mathbb{C} \mid \Re(\bar{\alpha}z + \beta) < 0 \}.$$

Show that all of the roots of the derivative $f'(z)$ lie in H .

Hint: Let $g(w) = f \left(\frac{\alpha(w-\beta)}{|\alpha|^2} \right)$. Show all of the roots of $g(z)$ have negative real parts and note that $f(z) = g(\bar{\alpha}z + \beta)$.

Remark: The result of this problem implies the Gauss-Lucas theorem: If $f(z)$ is a polynomial then all of the roots of $f'(z)$ are contained in the convex hull of the set of roots of $f(z)$. That is, any convex subset of \mathbb{C} which contains all of the roots of $f(z)$ also contains all the roots of $f'(z)$.

Problem 21. Let f be analytic in the disk $D(0, R)$. If $0 \leq r < R$ define

$$M(r) = \max_{|z|=r} |f(z)|.$$

Show that $|z| < r$ implies

$$\left| f^{(n)}(z) \right| \leq \frac{M(r)rn!}{(r-|z|)^{n+1}}.$$

In particular

$$\left| f^{(n)}(z) \right| \leq \frac{2M(2r)n!}{r^n} \quad \text{if } |z| \leq r \text{ and } 0 \leq 2r < R.$$

Problem 22. Let f be analytic in the disk $D(a, R)$. If $0 \leq r < R$ let $\gamma_r(t) = f(a + re^{it})$, $0 \leq t \leq 2\pi$. If $|\gamma_r|$ denotes the length of γ_r show that

$$|\gamma_r| \geq 2\pi r |f'(a)|.$$

Hint: Use Cauchy's integral formula applied to f' .

Problem 23. Let Ω be an open subset of \mathbb{C} and let f be analytic in Ω apart from a finite number of distinct poles a_1, a_2, \dots, a_n . If s_k is the singular part of the Laurent expansion of f at a_k then s_k is analytic in $\mathbb{C} \setminus \{a_k\}$ and $f - s_k$ has a removable singularity at a_k . Thus we can conclude that there is a function h analytic in Ω such that

$$f = s_1 + s_2 + \dots + s_n + h.$$

If $\Omega = \mathbb{C}$ and f is rational show that h is a polynomial. (This proves the existence of a partial fraction decomposition for any rational function.)

Problem 24. Show that

$$\int_0^{2\pi} (2 \cos(t))^{2n} dt = 2\pi \frac{(2n)!}{(n!)^2}.$$

Hint: Consider a suitable contour integral with integrand $\frac{1}{z} \left(z + \frac{1}{z}\right)^{2n}$.

Problem 25. Show that $f(x+iy) = x^2 - y^2$ has a derivative only at the origin (directly from the definition of the complex derivative).

Problem 26. Show that

$$\sec z = 1 + \sum_{n=1}^{\infty} \frac{E_{2n}}{(2n)!} (-1)^n z^{2n}$$

where the constants E_{2n} are the Euler numbers (apart from varying conventions concerning the sign). Find the radius of convergence without knowing anything about the E_{2n} . Then show the Euler numbers satisfy the relation

$$E_{2n} + \binom{2n}{2n-2} E_{2n-2} + \binom{2n}{2n-4} E_{2n-4} + \dots + \binom{2n}{2} E_2 + 1 = 0.$$

With our sign convention $(-1)^n E_{2n} > 0$. Show $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$, $E_{10} = -50521$. These numbers grow quite rapidly. For example,

$$E_{40} = 14851150718114980017877156781405826684425.$$

Problem 27. When viewed as a function of w

$$e^{z(w - \frac{1}{w})/2}$$

has an essential singularity at the origin. Thus we have a Laurent series

$$e^{z(w - \frac{1}{w})/2} = \sum_{n=-\infty}^{\infty} J_n(z) w^n.$$

The coefficient $J_n(z)$ is called the n^{th} Bessel function. Derive Bessel's formula (1824)

$$J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - z \sin \theta) d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin \theta) d\theta.$$

Problem 28. Let Ω be a bounded open set. Let f be a function continuous on the closure of Ω , that is, on $\Omega \cup \partial\Omega$. Suppose in addition that f is analytic in Ω . State the version of the maximum principle which applies to this situation.

Problem 29. Let Ω be a bounded open set. Let f be a function continuous on the closure of Ω , that is, on $\Omega \cup \partial\Omega$. Suppose in addition that f is analytic in Ω . Show (carefully) if f does not have a root in Ω then $|f|$ assumes its minimum value on $\partial\Omega$.

Problem 30. Let Ω be a bounded connected open set. Let f be a function continuous on the closure of Ω , that is, on $\Omega \cup \partial\Omega$. Suppose in addition that f is analytic in Ω . Suppose there is a constant c such that $|f(z)| = c$ for each $z \in \partial\Omega$. Show that f is constant or f has a root in Ω .

Problem 31. Let $\Omega = \{x + iy \mid -\pi/2 < y < \pi/2\}$ and let

$$f(z) = \exp(\exp z).$$

Show that $|f(z)| = 1$ for each $z \in \partial\Omega$, but f has no roots and f is not constant (in fact, not even bounded). Why does this example not contradict the result of the previous exercise?

5 Final Exam

Problem 32. Find the LAURENT expansion of $\frac{1}{(z-1)(z-3)}$ in the “annulus” A consisting of the set of points z with $1 < |z-2|$.

Problem 33. Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} 4^n z^{2n}.$$

Problem 34. If we expand $\sec z$ in a TAYLOR series about the origin,

$$\sec z = a_0 + a_1 z + a_2 z^2 + \cdots,$$

explain how we know that the series converges for $|z| < \frac{\pi}{2}$. Compute a_{1027} explicitly.

Problem 35. Let Ω be a bounded open set. Let f be a function continuous on the closure of Ω , that is, on $\Omega \cup \partial\Omega$. Suppose in addition that f is analytic in Ω . State the conclusion of a version of the maximum principle which applies to this situation. Now let g be continuous on the closure of Ω and analytic in Ω . Let $m > 0$ and suppose $|g(z)| \geq m$ for each $z \in \partial\Omega$. Show that $|g(z)| \geq m$ in Ω or g has a root in Ω .

Problem 36. Let f and g be analytic in Ω and let γ be a simple closed contour in Ω , null-homotopic in Ω . Suppose that f and g have at most a finite number of roots in Ω and that none of the roots lie on the trajectory of γ . Let $h(z) = f(z)/g(z)$ and let $\beta(t) = h(\gamma(t))$, so β is a closed contour in \mathbb{C} . Compute the winding number

$$\nu(\beta, 0) = \frac{1}{2\pi i} \int_{\beta} \frac{dw}{w}$$

and show that $\nu(\beta, 0) = 0$ if and only if f and g have the same number of roots inside γ . (**Hint:** Use the Principle of the Argument. Note you are proving part of ROUCHÉ’s theorem here.)

6 Contact Information

The contact information below is accurate as of Feb 15, 2001.

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