

# Advanced Engineering Mathematics – Mth 481

Archive – Summer 1993 Files

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This archive contains the sample problems and tests from Mth 481 Summer 1993. The original test instructions, headers and formatting have not been preserved.

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## 1 Sample Problems 1

**Problem 1.** Given  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all  $x$  and

$$f(x) = -x + \frac{x^2}{4} - \frac{x^3}{7} + \dots$$

for  $|x| < 1$  find the first four terms in the Taylor-Maclaurin series of  $e^{f(x)}$ .

**Problem 2.** Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^3}{5^n} x^{2n}.$$

**Problem 3.** Find the radius of convergence of the power series

$$(A) \quad \sum_{n=0}^{\infty} \frac{3^n n}{8^{n/2}} x^n \qquad (B) \quad \sum_{n=0}^{\infty} \frac{2^{2n+3} n^{3n}}{(3n)!} x^n.$$

**Problem 4.** Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^n}{n! 3^n} x^n.$$

**Problem 5.** Let  $y = \sum_{n=0}^{\infty} a_n x^n$  be a power series solution of the ordinary differential equation

$$(1 + 3x^2)y'' - x^2y' + xy = 0.$$

How large is the radius of convergence  $R$  of this power series guaranteed to be? (I.e. find a lower bound for  $R$ . Be sure to justify your answer.)

**Problem 6.** Find the first 10 terms in the power series solution to the initial value problem

$$y'' + x^2y = 0, \quad y(0) = k_1, \quad y'(0) = k_2$$

(i.e., all terms up through degree 9).

**Problem 7.** Find the first 10 terms in the power series solution to the initial value problem

$$y'' - x^3y = 0, \quad y(0) = k_1, \quad y'(0) = k_2$$

(i.e., all terms up through degree 9).

**Problem 8.** If we apply the method of FROBENIUS to find a fractional power series solution  $\sum_{n=0}^{\infty} a_n x^{n+\lambda}$ , ( $a_0 \neq 0$ ) to the ordinary differential equation

$$8x^2y'' + 10x(1-x)y' - 3y = 0$$

we end up with the indicial equation plus some recurrence relations for the coefficients  $a_n$ . Find the indicial equation and compute the indicial roots.

**Problem 9.** The ordinary differential equation

$$x^2y'' + xy' - (1 + x^2)y = 0$$

has a *regular singular point* at the origin. Find the indicial equation and the indicial roots. Then use the method of FROBENIUS to find a solution to the equation as a *fractional power series*. It is sufficient to give the first four nonzero terms in the series.

## 2 Sample Problems 2

**Problem 10.** Suppose we have a homogeneous linear ordinary differential equation with constant real coefficients and we determine it has characteristic roots

$$2 + 3i, 2 + 3i, 2 - 3i, 2 - 3i, -2, -2, -2, 1, 0, 0.$$

Here we have repeated each root according to its multiplicity. What is the general solution (in real form) of our differential equation?

**Problem 11.** Find the general solution of the linear ordinary differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = (x^2 + 2)e^{-x}.$$

**Problem 12.** Find the general solution of the linear ordinary differential equation

$$y'' - 4y' + 13y = e^{2x} \cos(3x).$$

**Problem 13.** Find a particular solution of the linear ordinary differential equation

$$y'' + y = \sec(x) \tan(x).$$

You might (or might not) find the following integrals useful:

$$\int \tan x \, dx = -\log |\cos x|, \quad \int \tan^2 x \, dx = -x + \tan x, \quad \int \sec x \, dx = \log |\sec x + \tan x|.$$

**Problem 14.** Consider the plane autonomous system

$$\frac{dx}{dt} = (x^2 - 1)y, \quad \frac{dy}{dt} = (y^2 - 1)x.$$

1. Find all of the equilibrium points.
2. At each equilibrium point  $(x_0, y_0)$  compute the Jacobi matrix  $J(x_0, y_0)$ .
3. Classify the equilibrium point at the origin.

By the Jacobi matrix we mean the coefficient matrix of the linearized system.

**Problem 15.** Compute  $e^{tA}$  where

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

**Problem 16.** Given

$$e^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

find a particular solution to the system of ordinary differential equations

$$\frac{d\vec{x}}{dt} = A\vec{x} + \begin{bmatrix} 1 \\ \cos(t) \end{bmatrix}.$$

**Problem 17.** Find the general solution (in real form)

$$\begin{aligned} \frac{dx_1}{dt} + \frac{dx_2}{dt} + 2x_1 - x_2 &= 0 \\ \frac{dx_1}{dt} - \frac{dx_2}{dt} + x_1 - 2x_2 &= 0 \end{aligned}$$

### 3 Test 1

**Problem 18.** Find the radius of convergence of the power series

$$(A) \sum_{n=0}^{\infty} \frac{3^n n^{3n}}{(2n)! n!} x^n \quad (B) \sum_{n=0}^{\infty} \frac{3^n n^{3n}}{(2n)! n!} x^{2n}.$$

**Problem 19.** Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

be a power series solution of the ordinary differential equation

$$\frac{d^2 y}{dx^2} + x^3 \frac{dy}{dx} + 3x^2 y = 0.$$

If we compute the coefficients  $a_n$  in the power series expansion of the solution  $y(x)$  we find that

$$y(x) = a_0 e^{-x^4/4} + a_1 y_1(x).$$

Find the power series representation of  $y_1(x)$ . Give enough terms to show the pattern, or give the  $n^{\text{th}}$  term.

**Problem 20.** The ordinary differential equation

$$9x^2 \frac{d^2 y}{dx^2} + 9x^2 \frac{dy}{dx} + 2y = 0$$

has a regular singular point at the origin. Find the indicial equation and the indicial roots. Then use the method of Frobenius to find linearly independent *fractional power series* solutions to the equation. (*Just a few terms, say four each.*)

**Problem 21.** For a certain fourth order linear ordinary differential equation with constant coefficients,

$$L[y] = xe^{-x} \cos(x) + x^2 e^{-x} + x$$

the characteristic polynomial is

$$(\lambda^2 + 2\lambda + 5)^2.$$

(A) Find the complimentary solution. (B) What form should the particular solution take, according to the method of undetermined coefficients? (*Do not compute the undetermined coefficients. Life's too short.*)

**Problem 22.** Solve the ordinary differential equation

$$9x^2 \frac{d^2 y}{dx^2} + 2y = 0.$$

## 4 Test 2

Consider the plane autonomous system

$$\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases}$$

where  $f$  and  $g$  are twice continuously differentiable. Let

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix}$$

be the Jacobi matrix. Suppose  $(x_0, y_0)$  is an isolated nondegenerate critical point of the system. In this case, near  $(x_0, y_0)$ , we may approximate the solutions of our system by

$$\begin{cases} x(t) \approx x_0 + u_1(t) \\ y(t) \approx y_0 + u_2(t) \end{cases}$$

where  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  is a solution of the *linearization* of our system at  $(x_0, y_0)$ :

$$\frac{d\vec{u}}{dt} = A\vec{u}, \quad A = J(x_0, y_0).$$

The following table gives the relation between the critical point of the linear system at the origin and the critical point of the original nonlinear system at  $(x_0, y_0)$ .

Linearization	Linearization	Nonlinear System	Stability
distinct real eigenvalues of the same sign	improper node	improper node	asymptotically stable or unstable
real eigenvalue of multiplicity 2, Jacobi matrix not diagonalizable	improper node	improper node	asymptotically stable or unstable
real eigenvalue of multiplicity 2, Jacobi matrix diagonalizable	proper node	proper node	asymptotically stable or unstable
real eigenvalues, opposite sign	saddle point	saddle point	unstable
complex conjugate eigenvalues, nonzero real part	focus	focus	asymptotically stable or unstable
pure imaginary eigenvalues	center	center	stable
		focus-center	stable
		focus	asymptotically stable or unstable

If the nonlinear system admits a first integral in a neighborhood of  $(x_0, y_0)$  then  $(x_0, y_0)$  is a *center* or a *saddle point*. There are no other possibilities in this case.

**Problem 23.** Compute the exponential  $e^{tA}$  given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}.$$

**Problem 24.** Given

$$e^{tA} = \frac{1}{7} \begin{bmatrix} 6e^{3t} + e^{-4t} & e^{3t} - e^{-4t} \\ 6e^{3t} - 6e^{-4t} & e^{3t} + 6e^{-4t} \end{bmatrix}$$

(A) solve the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x} + \begin{bmatrix} 1 \\ e^{3t} \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and (B) find the matrix  $A$ .

**Problem 25.** Consider the system

$$\begin{cases} \frac{dx}{dt} = x(1 - y^2) \\ \frac{dy}{dt} = (x + 1)(y + 2). \end{cases}$$

(A) Find all of the equilibrium points. (B) Classify the equilibrium points (to the extent possible) by studying the Jacobi matrix.

**Problem 26.** Suppose  $A^3 = 0$ . (A) Show  $e^{tA}$  is a polynomial in  $t$ . (B) Compute  $e^{tA}$  if

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

## 5 Contact Information

The contact information below is accurate as of Feb 15, 2001.

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