Consider the plane autonomous system

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

where f and g are twice continuously differentiable. Let

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{pmatrix}$$

be the Jacobi matrix. Suppose (x_0, y_0) is an isolated critical point of the system, that is, $f(x_0, y_0) = 0,\ \ g(x_0, y_0) = 0$. Suppose in addition this critical point is not degenerate, that is $\det J(x_0, y_0) \neq 0$. In this case we may write the solutions of our system in the form

$$x(t) = x_0 + u_1(t)$$
$$y(t) = y_0 + u_2(t)$$

where

$$\frac{du}{dt} = Au + R(u), \quad A = J(x_0, y_0) \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Here $R(u)$ consists of the remainder terms from the Taylor expansions of f and g. For small $|u|$ we have $|Au| \geq c|u|$ and $|R(u)| \leq c|u|^2$. Hence we expect that we can neglect $R(u)$. Thus we are led to the *linearization* of our system at (x_0, y_0)

$$\frac{du}{dt} = Au.$$

The following table gives the relation between the critical point of the linear system at the origin and the critical point of the original nonlinear system at (x_0, y_0).

Linearization	Linearization	Nonlinear System	Stabililty
distinct real eigenvalues of the same sign	improper node	improper node	asymptotically stable or unstable
real eigenvalue of multiplicity 2, Jacobi matrix not diagonizable	improper node	improper node	asymptotically stable or unstable
real eigenvalue of multiplicity 2, Jacobi matrix diagonizable	proper node	proper node	asymptotically stable or unstable
real eigenvalues, opposite sign	saddle point	saddle point	unstable
complex conjugate eigenvalues, nonzero real part	focus	focus	asymptotically stable or unstable
pure imaginary eigenvalues	center	center	stable
		focus-center	stable
		focus	asymptotically stable or unstable

If there is a neighborhood U of (x_0, y_0) and a function h continuously diffrentiable in U such that

$$f\frac{\partial h}{\partial x} + g\frac{\partial h}{\partial y} = 0 \text{ in } U$$

and

$$\left|\frac{\partial h}{\partial x}\right| + \left|\frac{\partial h}{\partial y}\right| \neq 0 \text{ in } U \text{ except possibly at } (x_0, y_0),$$

then (x_0, y_0) is a center or a saddle point for the nonlinear system. The function h is called a *first integral*. Physically it corresponds to a conserved quantity.

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