In this note we prove a quantitative version of the Fundamental Theorem of Algebra. The estimate we obtain is simple, has been known for a very long time, and is not very sharp. However even a bit of quantitative information makes the FTA more agreeable to an analyst!

We begin by reviewing a standard proof of the FTA. An immediate consequence of the Cauchy inequalities is

Theorem 1 (Liouville) *If f is an entire function and $|f|$ is bounded then f is constant.*

As a consequence we have

Theorem 2 *If f is an entire function and $|f(z)| \to \infty$ as $|z| \to \infty$ then f has at least one root.*

Indeed if f has no roots then $\frac{1}{f}$ is a bounded entire function.

Now let p be a nonconstant polynomial

$$p(z) = a_n z^n + \cdots + a_1 z + a_0, \quad a_n \neq 0.$$

For sufficiently large $|z|$ we have (by the triangle inequality)

$$|p(z)| \geq \frac{|a_n|}{2} |z|^n .$$

It follows that $|p(z)| \to \infty$ as $|z| \to \infty$ and so we have

Theorem 3 (Fundamental Theorem of Algebra, FTA) *A nonconstant polynomial has at least one root.*

This is a very elegant, if sparse statement. By invoking the division algorithm we see if p is a nonconstant polynomial,

$$p(z) = a_n z^n + \cdots + a_1 z + a_0, \quad a_n \neq 0,$$

then p has $m \geq 1$ distinct roots $z_1, z_2, \cdots, z_m$ and

$$p(z) = a_n \prod_{j=1}^{m} (z - z_j)^{k_j}$$

where $k_1 + \cdots + k_m = n$. Here k_j is called the multiplicity of the root z_j.

With a little extra work we can obtain a quantitative version of the FTA, that is, some information on where a root may be found. If $a \in \mathbb{C}$ define

$$r_p(a) = n \left| \frac{p(a)}{p'(a)} \right|$$

where it is understood that $r_p(a) = 0$ if $p(a) = 0$ (even if $p'(a) = 0$) and $r_p(a) = +\infty$ if $p'(a) = 0$ and $p(a) \neq 0$.

*Bent Petersen File ref: 611s2002˙fta.tex

Theorem 4 (Quantitative FTA) *If p is a nonconstant polynomial then for each $a \in \mathbb{C}$ the closed disk $\overline{D\,(a, r_p(a))}$ contains at least one root of p.*

For the proof, it suffices to consider the case $p(a) \neq 0$ and $p'(a) \neq 0$. If we apply the product rule to the complete factorization of p given above we obtain

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{m} \frac{k_j}{z - z_j}.$$

If we let $\mu = \min_{1 \leq j \leq m} |\, a - z_j \,|$ we have

$$\left| \frac{p'(a)}{p(a)} \right| \leq \sum_{j=1}^{m} \frac{k_j}{|\, a - z_j \,|} \leq \sum_{j=1}^{m} \frac{k_j}{\mu} = \frac{n}{\mu}.$$

Thus $\mu \leq \left| \frac{p(a)}{p'(a)} \right|$, that is, at least one of the roots $z_1, \cdots, z_m$ has distance at most $r_p(a)$ from a.

Example If $p(z) = z^5 - 4z^3 + z^2 + 1$ then $p(z)$ has at least one root in $\overline{D(1,1)}$. (In fact there are two roots in the disk. We will look at how to count roots later).

The log-derivative expression that we used above also may be used to obtain Lucas' theorem.

If $A \subset \mathbb{C}$ then the convex hull, $\mathrm{ch}(A)$, of A is the smallest convex set which contains A, that is, the intersection of all convex sets containing A. If A is finite, $A = \{z_1, \cdots, z_m\}$ then $\mathrm{ch}(A)$ consists of all points of the form $z = \lambda_1 z_1 + \cdots + \lambda_m z_m$, where $\lambda_j \geq 0$ for each j, and $\sum_{j=1}^{m} \lambda_j = 1$.

For a polynomial p we denote by $Z(p)$ the set of roots of p. Then

Theorem 5 (Lucas) *If p is a nonconstant polynomial then*

$$Z(p') \subseteq \mathrm{ch}(Z(p)).$$

If $p'(a) = 0$ then

$$0 = \frac{p'(a)}{p(a)} = \sum_{j=1}^{m} \frac{k_j}{a - z_j} = \sum_{j=1}^{m} \frac{k_j(\overline{a} - \overline{z_j})}{|\, a - z_j \,|^2}.$$

Thus

$$a = \sum_{j=1}^{m} \lambda_j z_j$$

where

$$\lambda_j = \frac{\frac{k_j}{|\, a - z_j \,|^2}}{\sum_{h=1}^{m} \frac{k_h}{|\, a - z_h \,|^2}}$$

The sets $Z(p)$ and $Z(p')$ are more "similar" than one might expect. For some examples see my Maple worksheet

Convex hull, Lucas theorem, Aziz's theorem and the Sendov-Ilyeff conjecture

on my web page at

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