Instruction: There are 10 problems below. Choose 3 problems to enjoy and for which to turn in solutions. If you turn in more than 3 be sure to indicate which three I should grade.

There may be some errors in the statements of the problems. Part of your job is to correct any such errors. As usual you may discuss the problems with anyone, but writing up your solutions should be a solitary activity.

Problem 1. Let $f : D(0,1) \to D(0,1)$ be analytic and suppose $f\left(-\frac{3}{5}\mathrm{i}\right) = 0$. Show that $\left|\, f\left(\frac{1}{5}\right)\right| < \frac{2}{3}$. (Actually a better bound is $\frac{5}{317}\sqrt{1585}$.)

Problem 2. Let $a, b \in \mathbb{C}$ be distinct points. Show that the family of circles passing through a and b is orthogonal to the circles of Apollonius.

Problem 3. Let Ω be a connected open subset of $\mathbb{C}$ and let f and g be analytic in Ω. Show that there exists an analytic function h on Ω such that $f = gh$ if and only if for each compact subset K of Ω there is a constant C_K such that

$$|\, f(z)\,| \le C_K \;|\, g(z)\,|\,, \quad \text{for each } z \in K.$$

Problem 4. Let T be a Möbius transform and let $a, b \in \mathbb{C}$ with $a \neq b$. Show that T has fixed points a and b if and only if there is a complex number λ such that

$$\frac{T(z) - a}{T(z) - b} = \lambda \frac{z - a}{z - b}, \quad z \in \mathbb{C}.$$

Conclude for each $n \in \mathbb{Z}$

$$\frac{T^n(z) - a}{T^n(z) - b} = \lambda^n \frac{z - a}{z - b}, \quad z \in \mathbb{C}$$

where T^n is the composition of T with itself n times if $n > 0$, T^n is the composition of the inverse of T with itself $-n$ times if $n < 0$ and T^0 is the identity map.

Problem 5. If f is analytic in $D(a, R)$, $0 < r < R$ and $\gamma(t) = a + r\mathrm{e}^{\mathrm{i}t}$ for $0 \le t \le 2\pi$ show that

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \overline{f(z)}\, \mathrm{d}z = r^2\, \overline{f'(a)}.$$

Problem 6. If $a \in \mathbb{C} \sim (-\infty, 0]$ then the "principal value" of a^z is

$$a^z = \exp(z \log a), \quad z \in \mathbb{C},$$

where $\log a$ is computed using the principal branch of the logarithm. In particular the principal value of e^z is $\exp(z)$, which is nice.

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Part A. Find the principal value of i^{i}.

Part B. Show that $a^{nz} = (a^z)^n$ (principal values) for any $n \in \mathbb{Z}$.

Part C. Show that a^{nz} and $(a^n)^z$ may differ by computing $(\mathrm{i}-1)^{2/3}$ and $\left((\mathrm{i}-1)^2\right)^{1/3}$

Problem 7. We know $z\sin(1/z)$ is continuous at the origin if we view it as a function of a real variable, however this is not the case if we view it as a function of a complex variable. Indeed, show $z\sin(1/z)$ has an essential singularity at 0. Compute the regular part, the singular part and the residue at the origin. How do you know that $z\sin(1/z)$ has a primitive in $\mathbb{C} \sim \{0\}$?

Problem 8. Suppose $F(x, y)$ is a rational function of two variables x and y with no singularities on the unit circle. Define

$$f(z) = F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2\mathrm{i}}\right).$$

Show that

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\ \mathrm{d}\theta = 2\pi\mathrm{i}\sum \mathfrak{R}es(f, a)$$

where the sum is over poles a of f with $|\, a \,| < 1$. Use this result to evaluate

$$\int_0^{2\pi} \frac{\cos(t)^2}{\cos(t)+\sin(t)+3}\ \mathrm{d}t.$$

Problem 9. For each z

$$w \longrightarrow \mathrm{e}^{z(w-\frac{1}{w})/2}$$

is analytic in $\mathbb{C} \sim 0$ and so has a Laurent series

$$\mathrm{e}^{z(w-\frac{1}{w})/2} = \sum_{n=-\infty}^{\infty} J_n(z)w^n.$$

Use what you know about the coefficients of a Laurent series to show

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}z\sin(t)-\mathrm{i}nt}\ \mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos(z\sin(t)-nt)\ \mathrm{d}t. \end{aligned}$$

(J_n is the n^{th} Bessel function, but you don't need to know this fact.)

Problem 10. Let $n \geq 1$ be an integer. Show

$$\int_0^{2\pi} \mathrm{e}^{\cos(\theta)} \cos\left(\sin(\theta) - n\theta\right)\ \mathrm{d}\theta = \frac{2\pi}{n!}.$$

Hint: Consider

$$\int_{|\, z \,|=1} \frac{\mathrm{e}^z}{z^{n+1}}\ \mathrm{d}z.$$
