We showed a function f analytic in an open disk $D(a,R)$, $R>0$, is the sum of a convergent power series

$$f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n$$

in the disk. Note it follows that the series has radius of convergence $\geq R$ and

$$c_n=\frac{1}{n!}\,f^{(n)}(a).$$

In the course of the proof we showed the coefficients are given by

$$c_n=\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}$$

where $\gamma(t)=a+r\mathrm{e}^{\mathrm{i}t}$ for any r with $0<r<R$. An immediate consequence is the Cauchy inequalities

$$|\,c_n\,|\leq \frac{M(r)}{r^n},\quad 0<r<R$$

where

$$M(r)=\max_{|\,z-a\,|=r}|\,f(z)\,|\,.$$

A consequence of the Cauchy inequalities is

Theorem 1 (Liouville) *If f is an entire function and there exist constants M, A, B and $q<1$ such that f satisfies one of the following inequalities*

1. $|\,f(z)\,|\leq M$
2. $|\,f(z)\,|\leq A+B\log(1+|\,z\,|)$
3. $|\,f(z)\,|\leq A+B\,|\,z\,|^q$

for each $z\in\mathbb{C}$, then f is constant.

The first two bounds are stronger than the last, so it suffices to show that the third inequality implies f is constant.

A similar argument shows

Theorem 2 *If f is an entire function and there exist constants A and m such that*

$$|\,f(z)\,|\leq A\,(1+|\,z\,|)^m\log(2+|\,z\,|)$$

for each $z\in\mathbb{C}$ then f is a polynomial of degree $\leq m$.

From Liouville's theorem we deduced the Fundamental Theorem of Algebra,

*Bent Petersen File ref: 611s2003˙sum2.tex

Theorem 3 (FTA) *Any nonconstant polynomial has a root.*

By invoking a bit of algebra we then showed that each polynomial factors into a product of a constant factor and factors of degree 1 (over $\mathbb{C}$).

We can also factor analytic functions to some extent by using the local power series representation. Thus we have

Theorem 4 *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be analytic. If $a \in \Omega$ and we define*

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

then g is analytic in Ω.

In particular if a is a root of f we have

$$f(z) = (z-a)g(z).$$

At this point, in the case of a root, it is natural to wonder how often we can divide. Here again the local power series representation comes to the rescue by giving us:

Lemma 5 *Let Ω be an open set in $\mathbb{C}$ and let f be a function analytic in Ω. If*

$$A = \left\{ a \in \Omega \ \mid \ f^{(n)}(a) = 0 \text{ for } n \geq 0 \right\}$$

then A is a relatively closed and open subset of Ω.

If Ω is connected then $A = \emptyset$ or $A = \Omega$. In the second case f is identically 0. We thus have

Lemma 6 *If Ω is a connected open set, f is a function analytic on Ω and f is not identically zero, then for each $z \in \Omega$ there exists and integer $n_z \geq 0$ such that*

$$f^{(n_z)}(z) \neq 0.$$

The facts about the set A above and a continuity argument yields

Theorem 7 (Identity Theorem) *Let Ω be a connected open set, let $f : \Omega \to \mathbb{C}$ be analytic, and let $(z_n)_{n\geq 1}$ be a sequence in Ω such that $f(z_n) = 0$ for $n \geq 1$. If $(z_n)_{n\geq 1}$ has an accumulation point in Ω then f is identically zero in Ω.*

If we apply the last lemma above to $f(z) - f(a)$ and make use of the local power series representation we see easily

Theorem 8 *Let Ω be a connected open set in $\mathbb{C}$ and let f be a nonconstant analytic function in Ω. Then for each $a \in \Omega$ there exists a unique integer $m \geq 1$ and a unique function g analytic in Ω such that $f(z) - f(a) = (z-a)^m g(z)$ and $g(a) \neq 0$. Moreover $g(a) = \frac{f^{(m)}(a)}{m!}$.*

Note it follows that the roots of f in Ω are isolated. This fact also follows more directly from the identity theorem above.

By finite induction we now obtain

Theorem 9 *Let Ω be a connected open set and let $f : \Omega \to \mathbb{C}$ be analytic. Suppose f has only finitely many roots in Ω, say $a_1, a_2, \cdots, a_n$. Then there exist unique positive integers $m_1, m_2, \cdots, m_n$ and a unique function g analytic in Ω such that $g(z) \neq 0$ for each $z \in \Omega$ and*

$$f(z) = (z - a_1)^{m_1} (z - a_2)^{m_2} \cdots (z - a_n)^{m_n} \ g(z)$$

for each $z \in \Omega$.

Note we call m_k the *multiplicity* of the root a_k.

Now let h be any function analytic in Ω and let f and g be as above. Then

$$h(z)\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{m_k h(z)}{z - a_k} + h(z)\frac{g'(z)}{g(z)}$$

for $z \in \Omega$. Note that the last term is analytic in Ω since $g(z) \neq 0$ for each $z \in \Omega$.

Now suppose that γ is a closed contour in Ω and suppose that Ω is a rectangle or a disk. Then by the Cauchy theorem and the Cauchy Integral theorem we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} h(z)\frac{f'(z)}{f(z)} \ \mathrm{d}z = \sum_{k=1}^{n} m_k \, \nu(\gamma, a_k) \, h(a_k).$$

As a special case if γ is a circle (traversed once) surrounding the roots we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} z^q \frac{f'(z)}{f(z)} \ \mathrm{d}z = \sum_{k=1}^{n} m_k \, a^q,$$

that is, the integral gives us the power sums of the roots. (This fact is important in connection with the Weierstrass Preparation theorem (Vorbereitungssatz) - see the notes on my web page.)

As a special case $q = 0$ gives us the number of roots, counted according to multiplicity, "inside" the circle γ,

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{f'(z)}{f(z)} \ \mathrm{d}z = \sum_{k=1}^{n} m_k.$$

As another special case consider when we have a circle surrounding a single root a of multiplicity 1. Then we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{z f'(z)}{f(z)} \ \mathrm{d}z = a.$$

Suppose instead of roots of $f(z)$ we look at roots $f^{-1}(w)$ of $f(z) - w$. Then we obtain (with some care)

$$f^{-1}(w) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{z f'(z)}{f(z) - w} \ \mathrm{d}z.$$

Since we can differentiate under the integral sign this expression allows to conclude that f^{-1} is analytic when f is one-to-one.

A consequence Cauchy inequalities and the FTA is

Theorem 10 *Let f be an entire function. Then f is a polynomial if and only if*

$$\lim_{|z|\to\infty} |f(z)|$$

exists. The limit is finite if and only if f is constant.

The Cauchy integral theorem instantly yields the Mean Value Theorem

Theorem 11 *Let Ω be an open subset of $\mathbb{C}$, let $a \in \Omega$ and let $R = \mathrm{dist}(a, \partial\Omega)$. If f is analytic in Ω and $0 < r < R$ then*

$$\begin{aligned} f(a) &= \frac{1}{2\pi}\int_0^{2\pi} f(a + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \\ f(a) &= \frac{1}{\pi r^2}\iint\limits_{D(a,r)} f(x + \mathrm{i}y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

An interesting consequence of the theorem above and the Cauchy–Schwarz inequality is,

Corollary 12 *For each function f analytic in Ω and each compact subset K of Ω we have*

$$\max_{z\in K} |f(z)| \le \frac{1}{\sqrt{\pi \mathrm{dist}(K,\Omega)}}\,\|f\|_2\,.$$

Thus convergence of a sequence of analytic functions in $\mathrm{L}^2(\Omega)$ implies uniform convergence on compact subsets of Ω. (This works for L^p-convergence as well.)

If f is analytic in $D(a, R)$ then

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z-a)^n$$

with normal convergence in $D(a, R)$. It follows

$$f(a + r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=0}^{\infty} c_n\,r^n \mathrm{e}^{\mathrm{i}n\theta}$$

and the convergence is uniform in θ (and in r for $0 \le r \le r_0 < R$). Note this is really a Fourier series. By making use of the uniform convergence we obtain

Theorem 13 (Gutzmer) *If f is analytic in $D(a, R)$ then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left| f(a + r\mathrm{e}^{\mathrm{i}\theta})\right|^2\,\mathrm{d}\theta = \sum_{n=0}^{\infty} |c_n|^2\,r^{2n}$$

where the c_n are the coefficients in the power series expanssion of f at a.

Note the Gutzmer equality immediately implies the Cauchy inequalities. Note also it is really a special case of the Parseval-Plancherel theorem for Fourier series.

Next week we will look at the maximum principle, open mapping principle and other properties of analytic functions. In particular we will obtain a very beautiful and simple proof of the maximum principle based on the Gutzmer equality.