Theorem 1 (Maximum Principle). *Let Ω be a connected open set in $\mathbb{C}$ and let f be analytic in Ω. If $|\, f \,|$ has a local maximum in Ω then f is constant in Ω.*

We gave two proofs – one based on the Mean Value Theorem and one based on Gutzmer's identity. The proof based on Gutzmer's identity goes as follows:

Proof. If $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ in $D(a,R)$, $0 < r < R$ and $|\, f(a) \,| \geq \left|\, f(a + r\mathrm{e}^{\mathrm{i}\theta}) \,\right|$ for $0 \leq \theta \leq 2\pi$ then

$$|\, c_0 \,|^2 = |\, f(a) \,|^2 \geq \frac{1}{2\pi} \int_0^{2\pi} \left|\, f(a + r\mathrm{e}^{\mathrm{i}\theta}) \,\right|^2 \, \mathrm{d}\theta = \sum_{n=0}^{\infty} |\, c_n \,|^2 \, r^{2n}$$

implies $c_n = 0$ for $n \geq 1$. $\square$

A very useful reformulation of the maximum principle is:

Theorem 2 (Maximum Principle). *Let Ω be a bounded open set in $\mathbb{C}$. Let f be a continuous function on $\overline{\Omega}$ and assume f is analytic in Ω. Then*

$$\sup_{z \in \Omega} |\, f(z) \,| = \max_{z \in \partial\Omega} |\, f(z) \,| \,.$$

If Ω is connected and there exists $z \in \Omega$ such that $|\, f(z) \,| = \max_{w \in \partial\Omega} |\, f(w) \,|$ then f is constant in Ω.

The function $f(z) = \mathrm{e}^{\mathrm{e}^z}$ in $\Omega = \left\{\, x + \mathrm{i}y \;\middle|\; |\, y \,| < \frac{\pi}{2} \,\right\}$ provides a counter-example in the case that Ω is unbounded. Later we will study some delicate maximum principles for unbounded domains.

Theorem 3. *Let Ω be a connected open subset of $\mathbb{C}$ and let f be a nonconstant analytic function in Ω. If $|\, f \,|$ has a local minimum at $a \in \Omega$ then $f(a) = 0$.*

Schwarz's lemma is a nice application of the maximum principle. It illustrates how we estimate a quotient by stepping away a bit from the roots of the denominator and the applying the maximum principle. We will use Schwarz's lemma later to characterize the automorphisms of the unit disk.

Theorem 4. *Suppose f is analytic in $D(a,R)$ where $R > 0$. If $f(0) = 0$ and $|\, f(z) \,| \leq M$ for $z \in D(a,R)$ then*

1. $|\, f(z) \,| \leq \dfrac{M}{R} |\, z \,|$ *for $z \in D(a,R)$, and*
2. $\left|\, f'(0) \,\right| \leq \dfrac{M}{R}$.

If we have equality in (1) for $z \neq 0$, or if we have equality in (2), then $f(z) = cz$ where $|\, c \,| = M/R$.

If $ad - bc \neq 0$ we define the Möbius transformation R by

$$R(z) = \frac{az + b}{cz + d}.$$

---

[*] Bent Petersen File ref: 611s2003˙sum3.tex

We regard R as mapping the Riemann sphere $\mathbb{C}_\infty$ (the one point compactification of $\mathbb{C}$) into itself by defining

$$R(\infty) = \begin{cases} \frac{a}{c} \text{ if } c \neq 0, \\ \infty \text{ if } c = 0. \end{cases}$$

and by defining $R(-d/c) = \infty$ if $c \neq 0$. It follows that R is a homeomorphism of the Riemann sphere. The inverse map is given by

$$R(w) = \frac{-dw + b}{cw - a}.$$

The Möbius transforms form a group $\mathbf{M}$ (under composition) of homeomorphisms of the Riemann sphere and the natural map

$$\Phi : GL(2, \mathbb{C}) \to \mathbf{M}$$

is a homorphism of groups with kernel

$$\ker(\Phi) = \{ \lambda I \;\mid\; \lambda \in \mathbb{C}, \lambda \neq 0 \}.$$

Each Möbius transformation is clearly the composition of (some of) homotheties (rotations and dilations), translations and inversion. Thus if we define "circle" to mean a circle or a line, then Möbius transfomations map "circles" to "circles." If $\alpha \in D(0,1)$ then

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}$$

is a Möbius transformation mapping the unit circle to itself. Since $\phi_\alpha(\alpha) = 0$ it is clear that ϕ_α is a homeomorphism of the unit disk $D(0,1)$ onto itself. Note

$$\phi_\alpha \circ \phi_\beta = \phi_{\frac{\alpha+\beta}{1+\alpha\overline{\beta}}}.$$

In particular, $\phi_\alpha^{-1} = \phi_{-\alpha}$.

By dividing by ϕ_α or composing with $\phi_{-\alpha}$ we obtain:

Theorem 5 (Schwarz). *Let $\alpha \in D(0,1)$, let f be analytic in $D(0,1)$ and suppose $|\, f(z) \,| \leq 1$ and $f(\alpha) = 0$. Then*

1. $|\, f(z) \,| \leq \left| \dfrac{z - \alpha}{1 - \overline{\alpha} z} \right|$ *for $z \in D(0,1)$, and*
2. $|\, f'(\alpha) \,| \leq \dfrac{1}{1 - |\, \alpha \,|^2}$.

If we have equality in (1) for some $z \neq \alpha$, or equality in (2), then $f = c\phi_\alpha$ for some c with $|\, c \,| = 1$.

> **Aside:** At this point we took a break from analysis and examined how the geometry of an open set enters into the questions of existence of a primitive (and Cauchy's theorem).
>
> Let Ω be a connected open set in $\mathbb{C}$ and let f be analytic in Ω. Let D_k be a sequence of disks with union Ω. Then f has a primitive F_k in D_k. If $D_j \cap D_k \neq \emptyset$ then $F_j - F_k$ is a constant b_{jk} in the connected set $D_j \cap D_k$. Clearly
>
> $$\begin{aligned} b_{jk} + b_{kj} &= 0 \text{ if } D_j \cap D_k \neq \emptyset, \\ b_{jk} + b_{kl} + b_{lj} &= 0 \text{ if } D_j \cap D_k \cap D_l \neq \emptyset, \end{aligned}$$

> that is, b is a Ĉech 1-cocycle. Suppose Ω has trivial 1-cohomology. Then we can find a 0-chain c with coboundary b. That is, there exist constants c_j such that
>
> $$b_{jk} = c_k - c_j, \text{ if } D_j \cap D_k \neq \emptyset.$$
>
> It follows that $F_j + c_j = F_k + c_k$ in $D_j \cap D_k$ for each j, k. Thus there exists a function F in Ω such that $F = F_k + c_k$ in D_k for each k. Clearly F is a primitive for f.
>
> I have simplified the discussion somewhat, but we are led to expect $H^1(\Omega, \mathbb{C}) = 0$ to be a sufficient condition for the existence of a primitive for any analytic function in Ω. (It is also necessary.)

Next we gave two proofs of the open mapping theorem:

Theorem 6. *Open Mapping Let Ω be a connected open subset of $\mathbb{C}$ and let f be a nonconstant analytic function in Ω. Then f is open, that is, for each open subset U of Ω, $f(U)$ is an open set.*

The proof in the text, due to Carathéodory, is an elegant and simple application of the minimum principle. We also gave a proof by root counting (principle of the argument). Suppose $f(z_0) = w_0$. Choose $r > 0$ so $D(z_0, r) \subset \Omega$. Let $\gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}$ and let $\beta = f \circ \gamma$. Then for each $w \notin$ traj (β) we have the number $n(w)$ of solutions z in $D(z_0, r)$ of the equation $f(z) = w$, counted according to multiplicity, is given by

$$n(w) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f'(z)}{f(z) - w} \, \mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int_\beta \frac{1}{u - w_0} \, \mathrm{d}u = \nu(\beta, w).$$

Since n is clearly continuous in $\mathbb{C} \sim$ traj (β) and is integer valued, it is constant on each component of $\mathbb{C} \sim$ traj (β). Thus $n(w) = n(w_0)$ in a neighborhood of w_0 which implies the open mapping theorem.

Next we discussed Morera's theorem:

Theorem 7 (Morera). *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a continuous function. Suppose*

$$\int_{\partial R} f(z) \, \mathrm{d}z = 0$$

for each (small) rectangle R contained in Ω and with edges parallel to the coordinate axes. The f is analytic in Ω.

The proof ie wasy: The hypotheses imply f has a primitive (which must be analytic since it is differentiable). But then the local power series representation of the primitive, and the fact that we can differentiate convergent power series term by term, imply f has local power series representations and so is analytic.

Corollary 8. *Let Ω be a an open subset of $\mathbb{C}$ and let $(f_n)_{n \geq 1}$ be a sequence of analytic functions in Ω. If this sequence is uniformly Cauchy on compacta in Ω then $\lim_{n\to\infty} f_n(z) = f(z)$ exists for each $z \in \Omega$ and defines a function f in Ω. Moreover f is analytic and $\lim_{n\to\infty} f_n = f$ uniformly on compacta in Ω.*

Now let $A(\Omega)$ be the space of functions analytic on Ω. Let K_n be a sequence of compact subsets of Ω with interiors with union Ω. Define

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\max_{z \in K_n} |\, f(z) - g(z) \,|}{1 + \max_{z \in K_n} |\, f(z) - g(z) \,|}$$

for $f, g \in A(\Omega)$. Then d is a complete metric on $A(\Omega)$ and d induces the topology of uniform convergence on compacta in Ω. Let $L^2(\Omega)$ be the space of square-integrable functions on Ω. Recall we showed (from the mean value theorem) for each compact subset K of Ω there is a constant $C_K > 0$ such that

$$\max_{z \in K} |\, f(z) \,| \leq C_K \, \|f\|_2 \,, \quad f \in A(\Omega) >$$

In particular, for each $z \in \Omega$ $f \to f(z)$ is an L^2 continuous linear functional on $A(\Omega) \cap L^2(\Omega)$.

Corollary 9. *$A(\Omega) \cap L^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and so is a Hilbert space in the L^2 inner product.*