Let Ω be a connected set, let f be a function analytic in Ω and suppose $f(z) \neq 0$ for each $z \in \Omega$. Then g is called a branch of the logarithm of f, or, less formally, a branch of $\log f(z)$, if

1. g is analytic in Ω
2. $\mathrm{e}^{g(z)} = f(z)$ for each $z \in \Omega$.

Actually, it suffices to assume continuity of g – analyticity then follows.

We write $g(z) = \log f(z)$ to indicate that g is a branch of $\log f(z)$. Be cautious. It's misleading to think of $\log f(z)$ as a simple composition. Moreover the usual properties of logarithms do not hold without additional hypotheses (the problem is in how the argument is determined).

We have not discussed simply connected sets yet, but we will shortly. For now it suffices to know if Ω is a simpply connected open subset of $\mathbb{C}$ and γ is a closed contour then

$$\int_\gamma f(z)\ \mathrm{d}z = 0$$

for each function f analytic in Ω. We have already seen that open sets which are star-shaped at a point, or less generally are convex, do in fact satisfy this condition. We will see shortly that star-shaped sets are in fact simply connected.

Theorem 1. *If Ω is a simply connected open set, $f : \Omega \to \mathbb{C}$ is analytic, $f(z) \neq 0$ for each $z \in \Omega$, $z_0 \in \Omega$, $\mathrm{e}^{w_0} = f(z_0)$ and for each $z \in \Omega$, γ_z is a contour joining z_0 to z, the*

$$g(z) = w_0 + \int_{\gamma_z} \frac{f'(u)}{f(u)}\ \mathrm{d}u$$

define a branch of $\log f(z)$ with $g(z_0) = w_0$. Moreover if h is a branch of $\log f(z)$ with $h(z_0) = w_0$ then $h(z) = g(z)$ for each $z \in \Omega$.

We proved in class that g is a branch of $\log f$. For the last part note that $g(z) - h(z) = 2\pi \mathrm{i} k(z)$ where k is integer valued and continuous (since $g - h$ is continuous). Since Ω is connected k is constant. But $k(z_0) = 0$.

Corollary 2. *If g is a branch of $\log f$ in an open set Ω then $g'(z) = \frac{f'(z)}{f(z)}$ for $z \in \Omega$.*

Corollary 3. *If Ω is a simply connected open set then for each $a \notin \Omega$ there is a branch of $\log(z - a)$ in Ω. In particular $\frac{1}{z-a}$ has a primitive in Ω.*

An immediate and important consequence of the last corollary above is

Corollary 4. *If Ω is a simply connected open set in $\mathbb{C}$ and γ is a closed contour in Ω then for each $a \notin \Omega$ we have*

$$\nu(\gamma, a) = 0.$$

*Bent Petersen File ref: 611s2003˙sum4.tex

The principal branch of $\log z$ is defined as follows. The domain is

$$\Omega = \{ z \in \mathbb{C} \ \mid \ z \notin (-\infty, 0] \} .$$

Note that Ω is simply connected (since it is star-shaped). For each $z \in \Omega$ we have a unique argument θ such that

$$z = r\mathrm{e}^{\mathrm{i}\theta}, \quad r = |\, z \,|, \quad |\, \theta \,| < \pi.$$

We define

$$\log z = \log r + \mathrm{i}\theta$$

where $\log r$ denotes the usual logarithm of the positive real number r.

Application

The complex logarithm comes up even in real analysis. As an example, suppose we are interested in

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{x} \ \mathrm{d}x$$

where ϕ is, for example, continuously differentiable and vanishes outside a compact set. This integral in general is divergent, so we study instead the principal value

$$\lim_{\epsilon \downarrow 0} \int_{|\, x \,| > \epsilon} \frac{\phi(x)}{x} \ \mathrm{d}x,$$

which is easily seen to exist (for example, by integration by parts – see below).

Another approach to "regularizing" the divergent integral is to step into the complex plane and to consider

$$\lim_{y \downarrow 0} \int_{-\infty}^{\infty} \frac{\phi(x)}{x + \mathrm{i}y} \ \mathrm{d}x.$$

For $y > 0$ integration by parts yields

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{x + \mathrm{i}y} \ \mathrm{d}x = -\int_{-\infty}^{\infty} \log(x + \mathrm{i}y)\phi'(x) \ \mathrm{d}x$$

where we use the principal branch of the logarithm. Now

$$\lim_{y \downarrow 0} \log(x + \mathrm{i}y) = \begin{cases} \log |\, x \,| & \text{if } x > 0 \\ \mathrm{i}\pi + \log |\, x \,| & \text{if } x < 0 \end{cases}$$

By dominated convergence (using $\log |\, x \,|$ is locally integrable) we obtain

$$\begin{aligned} \lim_{y \downarrow 0} \int_{-\infty}^{\infty} \log(x + \mathrm{i}y)\phi'(x) \ \mathrm{d}x &= \mathrm{i}\pi \int_{-\infty}^{0} \phi'(x) \ \mathrm{d}x + \int_{-\infty}^{\infty} \log |\, x \,| \, \phi'(x) \ \mathrm{d}x \\ &= \mathrm{i}\pi\phi(0) + \lim_{\epsilon \downarrow 0} \int_{|\, x \,| > \epsilon} \log |\, x \,| \, \phi'(x) \ \mathrm{d}x \end{aligned}$$

By integration by parts the last integral is

$$(\phi(-\epsilon) - \phi(\epsilon)) \log \epsilon - \int_{|\, x \,| > \epsilon} \frac{\phi(x)}{x} \ \mathrm{d}(x).$$

Since $\epsilon \log \epsilon \to 0$ as $\epsilon \downarrow 0$ the first term goes to 0. Thus we have

$$\lim_{y\downarrow 0} \int_{-\infty}^{\infty} \frac{\phi(x)}{x+\mathrm{i}y}\, \mathrm{d}x = -\mathrm{i}\pi\phi(0) + \lim_{\epsilon\downarrow 0} \int_{|\,x\,|>\epsilon} \frac{\phi(x)}{x}\, \mathrm{d}(x).$$

In distribution notation we have

$$\lim_{y\downarrow 0} \left(\frac{1}{x+\mathrm{i}y}\right) = -\mathrm{i}\pi\delta + \mathrm{Pv}\left(\frac{1}{x}\right).$$

This equation is called Dirac's equation. Here δ denotes Dirac's delta. Note the same argument shows

$$\lim_{y\downarrow 0} \left(\frac{1}{x-\mathrm{i}y}\right) = \mathrm{i}\pi\delta + \mathrm{Pv}\left(\frac{1}{x}\right).$$

which yields (by subtraction)

$$\lim_{y\downarrow 0} \frac{1}{\pi}\, \frac{y}{x^2+y^2} = \delta.$$

This is the Poisson equation for the upper half plane. It means

$$\lim_{y\downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\phi(x)}{x^2+y^2}\, \mathrm{d}x = \phi(0)$$

for each continuously differentiable function ϕ with compact support.

Removable Singularities

As a consequence of Morera's theorem we obtain

Theorem 5. *Le Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be continuous. Let $L \subset \Omega$ be a closed line segment. Suppose f is analytic in $\Omega \sim L$. Then f is analytic in Ω.*

If we apply the previous result to the short segment $\{a\}$ and we use the fact that an analytic function which vanishes at a point a has an analytic quotient when divided by $z - a$, we obtain

Theorem 6 (Riemann Removable Singularity Theorem). *If $f : D(a,r) - \{a\} \to \mathbb{C}$ is analytic and*

$$\lim_{z\to a} (z-a)f(z) = 0$$

then f continues uniquely to an analytic function in $D(a,r)$.

The phrase "continues to" is the traditional way of saying "extends to" in complex analysis.

Corollary 7. *If $f : D(a,r) \sim \{a\} \to \mathbb{C}$ is analytic and $|\,f\,|$ is bounded on $D(a,r) \sim \{a\}$, then f continues uniquely to an analytic function on $D(a,r)$.*

Theorem 8 (Schwarz Reflection Theorem). *Let $\Pi^+ = \{\, z \in \mathbb{C} \mid \Im(z) > 0 \,\}$ be the open upper half plane. Let Ω be an open subset of Π^+. Let L be a line segment on the real line $\mathbb{R}$. Assume for each $a \in L$ there is $r_a > 0$ such that*

$$D(a, r_a) \cap \Pi^+ \subset \Omega.$$

Let $\Omega^* = \{\, z \in \mathbb{C} \mid \overline{z} \in \Omega \,\}$. Then

$$\Omega_1 = \Omega \cup L \cup \Omega^*$$

is an open subset of $\mathbb{C}$. Suppose f is analytic in Ω and there exists a real valued function g such that

$$\lim_{y\downarrow 0} f(x+\mathrm{i}y) = g(x)$$

uniformly for x in compacta in L. The there exists F analytic in Ω such that

1. $\overline{F(z)} = F(\overline{z})$ *for* $z \in \Omega_1$.
2. $F(z) = f(z)$ *for* $z \in \Omega$.

Actually it suffices to assume for each sequence z_n with $z_n \to a \in L$ we have $\Im\mathfrak{m}(f(z)) \to 0$. This fact is harder to prove and requires some knowledge of harmonic functions.

Simply Connected Sets

The definition of a simply connected set is subtle: a connected open set Ω is simply connected if for each $a \notin \Omega$ and each $\epsilon > 0$ there is a continuous map $\gamma\colon [0,\infty) \to \mathbb{C}$ such that

1. $\gamma(0) = a$
2. dist($\gamma(t), \mathbb{C} \sim \Omega$)$< \epsilon$ for all $t \geq 0$
3. $\lim_{t\to\infty} \gamma(t) = \infty$.

An example of a simply connected open set whose complement is not arc-connected is given in our text.

Theorem 9. *Let Ω be a simply connected open set in $\mathbb{C}$ and let K be a compact subset of $\mathbb{C}$. If $\partial K \subset \Omega$ then $K \subset \Omega$.*

Theorem 10. *If Ω is an open set star-shaped at some point $c \in \Omega$ then Ω is simply connected.*

Corollary 11. *If Ω is a convex open set then Ω is a simply connected set.*

Cauchy Theorem

Now we give a bootstrap argument to obtain a fairly useful version of the Cauchy theorem.

Lemma 12. *Let Ω be a simply connected open set and let γ be a simple closed piecewise linear path in Ω consisting of only horizontal and vertical segments. Then*

$$\int_\gamma f(z)\ \mathrm{d}z = 0$$

for each f analytic in Ω.

Lemma 13. *If f is analytic in a simply connected open set Ω then f has a primitive in Ω.*

Theorem 14 (Cauchy Theorem for simply connected sets). *Let Ω be a simply connected open set and let f be analytic in Ω. If γ is a closed contour in Ω then*

$$\int_\gamma f(z)\ \mathrm{d}z = 0.$$

Note how the proof goes: first we prove a simple version of the Cauchy theorem for a very restrictive class of closed contours. This result is good enough to obtain a primitive for each analytic function in Ω. But then the Cauchy theorem for an arbitrary closed contour follows by the Fundamental Theorem of Calculus.

There are other versions of the Cauchy theorem, for example, Ω may be an arbitrary open set in which case we may require that γ be null-homotopic (or, for a stronger theorem, null-homologous).

Isolated Singularities

If f is analytic in $D(a,r) \sim \{a\}$ where $r > 0$ then we say f has an isolated singularity at a. We classified the isolated singularities as removable singularities, polar singularities (poles) and essential singularities.

The isolated singularity a is removable if there is an analytic function g in $D(a,r)$ such that $g_{|D(a,r)\sim\{a\}} = f$. A sufficient condition is $|\, f \,|$ be bounded in a neighborhood of a.

The isolated singularity a is a pole of order $m > 0$ if $(z-a)^m f(z)$ has a removable singularity at a but $(z-a)^{m-1} f(z)$ does not.

The isolated singularity a is essential if it is not removable and is not a pole.

Theorem 15 (Casorati-Weierstrass). *Let Ω be an open set, let $a \in \Omega$ and let $f : \Omega \sim \{a\} \to \mathbb{C}$ be analytic. If a is an essential singularity for f then the image $f(\Omega \sim \{a\})$ is dense in $\mathbb{C}$.*

Theorem 16. *If f has an isolated singularity at a then a is a pole for f if and only if $\lim_{z\to a} |\, f \,| = \infty$.*

Laurent Decomposition and Laurent Series

Let $0 \le r_1 < r_2 \le \infty$. Then we define the annulus

$$D(a, r_1, r_2) = \{\, z \in \mathbb{C} \;\;|\;\; r_1 < |\, z - a \,| < r_2 \,\}\,.$$

Note that $D(a, 0, r)$ is just the punctured disk $D(a,r) \sim \{a\}$.

Theorem 17 (Laurent Decomposition). *Let f be analytic in the annulus $D(a, r_1, r_2)$. Then there exist unique functions f_1 and f_2 such that*

1. *f_1 is analytic in $D(0, r_2)$,*
2. *f_2 is analytic in $D(0, 1/r_1)$,*
3. *$f_2(0) = 0$, and*
4. *$f(z) = f_1(z-a) + f_2((z-a)^{-1})$ for $z \in D(a, r_1, r_2)$.*

Note $f(z-a)$ is called the regular part of f and $f_2(1/(z-a))$ is called the principal part.

Theorem 18 (Laurent Series). *Let f be analytic in the annulus $D(a, r_1, r_2)$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

with normal convergence in $D(a, r_1, r_2)$. This series is unique. If $r_1 < r < r_2$ then

$$\begin{aligned} c_n &= \frac{1}{2\pi \mathrm{i}} \int_{|\, z-a \,|=r} \frac{f(z)}{(z-a)^{n+1}}\ \mathrm{d}z \\ &= \frac{r^{-n}}{2\pi} \int_0^{2\pi} f(a + r\mathrm{e}^{-\mathrm{i}n\theta})\ \mathrm{d}\theta. \end{aligned}$$

for any $n \in \mathbb{Z}$. Moreover we have the Gutzman equality

$$\frac{1}{2\pi} \int_0^{2\pi} \left|\, f(a + r\mathrm{e}^{\mathrm{i}\theta}) \,\right|^2\ \mathrm{d}\theta = \sum_{n=-\infty}^{\infty} |\, c_n \,|^2\, r^{2n}.$$

Corollary 19 (Cauchy inequalities). *With the hypotheses of the previous theorem if*

$$M(r) = \max_{|\,z-a\,|=r} |\, f(z)\,|$$

we have

$$|\, c_n \,| \leq \frac{M(r)}{r^n}, \quad r_1 < r < r_2, \quad n \in \mathbb{Z}.$$

The proof of the Laurent series expansion depends on the lemma:

Lemma 20. *If f is analytic in $D(a, r_1, r_2)$ then the integral*

$$\int_{|\,z-a\,|=r} f(z)\ \mathrm{d}z$$

is independent of r for $r_1 < r < r_2$.

With a single exception each term in the Laurent series has a primitive. Since we have uniform convergence on compact subsets of $D(a, r_1, r_2)$ it follows that if γ is a closed contour in $D(a, r_1, r_2)$ then

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma f(z)\ \mathrm{d}z = c_{-1}\, \nu(\gamma, a).$$

Theorem 21. *If f is analytic in $D(a, r_1, r_2)$ then there is a unique constant b such that*

$$f(z) - \frac{b}{z-a}$$

has a primitive in $D(a, r_1, r_2)$. Moreover

$$b = c_{-1}$$

where c_n is the n^{th} Laurent coefficient of f relative to $D(a, r_1, r_2)$.

We call c_{-1} the *residue* of f relative to $D(a, r_1, r_2)$ and write $\mathfrak{R}es(f, D(a, r_1, r_2)$ for it. In case $r_1 = 0$ we speak of the residue of f at a (since it does not depend on r_2) and write $\mathfrak{R}es(f, a)$ for it.

Another way to prove the theorem above is to to differentiate a suitable normally convergent series term by term. That we can do so follows from

Lemma 22. *Let Ω be an open subset of $\mathbb{C}$. Let f_n be a sequence of functions analytic in Ω and suppose $f_n \to f$ uniformly on compact subsets of Ω. Then f is analytic on Ω and $f_n' \to f'$ uniformly on compact subsets of Ω.*

Now consider f analytic in $D(a, r) \sim \{a\}$, so

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

with uniform convergence on compacta in $D(a, r) \sim \{a\}$.

- f has a removable singularity at a if and only if $c_n = 0$ for $n < 0$.
- f has a pole of order $m > 0$ at a if and only if $c_n = 0$ for $n < -m$ and $c_{-m} \neq 0$.

- f has an essential singularity at a if and only if $c_n \neq 0$ for some sequence $n \downarrow -\infty$.

Suppose now that f has a pole of order $\leq m$ at a. Then clearly

$$\mathfrak{Res}(f,a) = \lim_{z\to a} \frac{1}{(m-1)!} \left(\frac{d}{dz}\right)^{m-1} \left[(z-a)^m f(z)\right].$$

For a pole of order ≤ 1 this simplifies to

$$\mathfrak{Res}(f,a) = \lim_{z\to a}(z-a)f(z).$$

For an essential singularity we do not have a convenient formula for the residue.

Example computing a simple Laurent expansion

Consider

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

To expand $f(z)$ in a Laurent series in $1 < |\,z\,| < 2$, for example, we expand each term above:

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n, \quad \text{for } |\,z\,| < 2,$$

and

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}, \quad \text{for } |\,z\,| > 1.$$

Thus

$$c_n = \begin{cases} -2^{-n-1} & \text{if } n \geq 0 \\ -1 & \text{if } n < 0. \end{cases}$$

Examples computing residues

Suppose f and g are analytic in $D(a,r)$, f has a root of multiplicity n at a, g has a root of multiplicity m at a and $m > n$. Then f/g has a pole of order $m-n$ at a and

$$\mathfrak{Res}\left(\frac{f}{g},a\right) = \lim_{z\to a} \frac{1}{(m-n-1)!} \left(\frac{d}{dz}\right)^{m-n-1} \left[(z-a)^{m-n}\frac{f(z)}{g(z)}\right].$$

$$\begin{aligned}
&m = 1, n = 0,\\
&\mathfrak{Res}\left(\tfrac{f}{g},a\right) &&= \frac{f(a)}{g'(a)}\\
&m = 2, n = 0,\\
&\mathfrak{Res}\left(\tfrac{f}{g},a\right) &&= \frac{2}{3}\,\frac{3f'(a)g''(a) - f(a)g'''(a)}{(g''(a))^2}\\
&m = 2, n = 1,\\
&\mathfrak{Res}\left(\tfrac{f}{g},a\right) &&= \frac{2f'(a)}{g''(a)}\\
&m = 3, n = 0,\\
&\mathfrak{Res}\left(\tfrac{f}{g},a\right) &&= -\frac{3}{40}\,\frac{-40f''(a)g'''(a)^2 + 4f(a)g^{(5)}(a)g'''(a) + 20f'(a)g'''(a)g^{(4)}(a) - 5f(a)g^{(4)}(a)^2}{g'''(a)^3}\\
&m = 3, n = 2,\\
&\mathfrak{Res}\left(\tfrac{f}{g},a\right) &&= 3\frac{f''(a)}{g'''(a)}
\end{aligned}$$

Try working out a few yourself (or check the ones above).

Winding Numbers

If γ is a closed contour and $a \notin$ traj(γ) then

$$\nu(\gamma, a) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{\mathrm{d}z}{z - a}$$

is the winding number of γ about a or the index of γ at a. We have made use of it several times without knowing much about it.

Theorem 23. *$\nu(\gamma, a)$ is an integer.*

Note from the definition that $\nu(\gamma, a)$ depends continuously on a for $a \notin$ traj(γ). Since it is integer valued it must be constant on each component of $\mathbb{C} \sim$ traj(γ). Since the trajectory of γ is compact it must lie inside some disk. The complement of the disk is connected and so lies in a component of $\mathbb{C} \sim$ traj(γ). All other components lie in the disk and so are bounded. By considering a tangent line to the disk we obtain a half plane that does not intersect the trajectory of γ. For a in this half plane there is branch of $\log(z - a)$ in the complementary half plane containing the trajectory of γ. Thus $\nu(\gamma, a) = 0$. Since the winding number is constant on components we have

Theorem 24. *If γ is a closed contour in $\mathbb{C}$ there is a unique unbounded component of $\mathbb{C} \sim$ traj(γ). For a in this unbounded component we have $\nu(\gamma, a) = 0$.*

Theorem 25 (Jordan). *The complement of a simple closed curve in the plane has two components.*

By the discussion above one component is bounded and the other is not.

A simple closed contour γ parametrized so that $\nu(\gamma, a) = 0, 1$ for each a not on the trajectory, is said to be *regular*. We call $\{ a \notin \text{traj}(\gamma) \mid \nu(\gamma, a) = 1 \}$ the *inside* of γ. The inside component obviously is the unique bounded component. The unbounded component we call the *outside* of γ (even if γ is not simple).

Residue Theorem

Theorem 26 (Residue Theorem). *Let Ω be a simply connected open set in $\mathbb{C}$ and let f be analytic in Ω apart from a finite number of distinct isolated singularities $a_1, a_2, \cdots, a_m$. Let γ be a closed contour in $\Omega \sim \{a_1, a_2, \cdots, a_m\}$. Then*

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} f(z)\ \mathrm{d}z = \sum_{k=1}^{m} \nu(\gamma, a_k)\, \mathfrak{R}es(f, a_k).$$

If γ simple regular closed contour then

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} f(z)\ \mathrm{d}z = \sum \mathfrak{R}es(f, a_k)$$

where the sum is over a_k inside γ.

There are more general versions of the residue theorem.