Mth 611 Complex Analysis
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Assignment 1
Cardano's Method
2 Problems. Due April 5.

You should work through the first chapter of our text as a review of complex numbers. This assignment is very simple. It requires only a small amount of complex arithmetic.

Scipione dal Ferro solved some cubic equations around 1500 and confided his method to Antonio Maria Fior around 1510. In 1535 Fior, who must have been playing with dal Ferro's method, challenged Tartaglia (Niccoló Fontana) to solve 30 cubics. Tartaglia did so. In 1539 Gerolamo Cardano pressed Tartaglia to reveal his method. Cardano determined that Tartaglia's method was the same as dal Ferro's and in 1542 he published it in spite of his promise to keep it secret. Cardano's publication includes a detailed proof of the method in numerous special cases. The modern version of the Ferro-Tartaglia-Cardano method is usually called Cardano's method. The quartic equation was solved by Cardano's student, Lodovico Ferrari, and was also published by Cardano.

Let's have a look at Cardano's method (in modern notation):

Consider the equation

$$x^3 + ax^2 + bx + c = 0.$$

If we substitute $x = y - a/3$ we obtain

$$y^3 + py + q = 0$$

where

$$p = b - \frac{a^2}{3} \qquad \text{and} \qquad q = c - \frac{ab}{3} + \frac{2a^3}{27}.$$

Now substitute $y = u - p/(3u)$, multiply by u^3 and then substitute $v = u^3$ to obtain the quadratic

$$v^2 + qv - \frac{p^3}{27} = 0.$$

We compute the roots of the quadratic, and for each of those, three cubic roots, etc. There are some relations between our solutions, so in fact our 6 roots coalesce to just 3.

The discriminant of the quadratic above is

$$\Delta = q^2 + \frac{4p^3}{27}.$$

If we take the product of the values of the polynomial $y^3 + py + q = 0$ at its critical points we just obtain the discriminant. In the case of a real cubic with 3 real roots these critical points must be a positive local maximum and a negative local minimum (draw a picture), and so the discriminant must be negative.

It follows for a real cubic with 3 real roots in Cardano's method we must deal with a quadratic with complex roots and take cubic roots of these complex numbers! Cardano actually observed this fact (calling the cubic "irreducible" in this case), but he missed the opportunity to follow up and to make a serious study of complex numbers. Curiously Cardano also regarded negative solutions as "fictitious" but gave the negative solutions in every case, though he missed the opportunity to put forward negative solutions as real.

The case of positive discriminant corresponds to the real cubic having exactly one real root. You might enjoy proving this fact and also determining what happens when the discriminant is 0.

Around 1590 François Vieta discovered a trigonometric approach to finding roots of cubics. This method eliminates the intermediate complex numbers. Vieta also did something much more important - he introduced the idea of symbolic coefficients for polynomials.

Problem 1. Show that cubic $x^3 + 2x + 1$ has one real root and use Cardano's method to compute the root (exactly).

Problem 2. Use Cardano's method to find the three real roots of $x^3 - 28x + 48$. (This will entail some complex arithmetic.)

Note in the second problem the roots are integers and are pretty easy to find by simple methods. That isn't the point. The point is to use Cardano's method and to observe that complex numbers occur in the calculation. Note Cardano would have written the equation as $x^3 + 48 = 28x$ to avoid the negative coefficient. This restriction leads to many special cases. In Vieta's work the symbolic coefficients were still regarded as positive numbers, but who is going to check every newly calculated coefficient? Symbolic calculations hid the negative numbers and therefore made them gradually more acceptable, though Vieta discarded negative solutions entirely. The restriction to positive numbers was due to viewing numbers as geometric quantities (though even in geometry the notion of orientation can lead to negative numbers).