Mth 611 Complex Analysis
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Assignment 2
Complex Numbers
10 Problems. Due April 9.

As pointed out in Assignment 1, complex numbers were forced on Cardano in his study of the roots of cubics, published in *Ars Magna*, 1545. Cardano ignored complex roots, but for a cubic with 3 distinct real roots, complex numbers occur in Cardano's calculation of the real roots. Cardano regarded them as meaningless - just symbols.

Complex numbers were used more and more from the 1750's on by, for example, d'Alembert, Laplace and others, and especially by Euler. However, even at this time, complex numbers were regarded as part of the technique rather than as "real" objects.

The interpretation of complex numbers as vectors in the plane is due to Wessel (1797) and Argand (1806), and as points in the plane by Gauss (1831 and earlier). These interpretations gave geometric substance to complex numbers and made them more acceptable.

In the period 1814-1846 Cauchy developed a full-blown theory of analytic functions of a complex variable. There was no longer any doubt about complex numbers being as real as any other mathematical objects.

This assignment contains a few more complex arithmetic problems. I am assuming you have worked through chapter 1 of our text. Note in particular

$$\operatorname{cis}\theta = \cos\theta + \mathrm{i}\sin\theta$$

in section 4. After we discuss the complex exponential we will be able to replace $\operatorname{cis}\theta$ by $\mathrm{e}^{\mathrm{i}\theta}$. Out of habit and preference I will use the exponential notation below even though we have not justified it yet.

The critical property of cis is de Moivre's formula, or the exponential group property:

$$\operatorname{cis} n\theta = (\operatorname{cis}\theta)^n$$

for any integer n, or, more generally

$$\operatorname{cis}(\theta+\phi) = \operatorname{cis}\theta\operatorname{cis}\phi.$$

Problem 1. Given

$$\cos\frac{\pi}{24} = \frac{1}{4}\sqrt{8+2\sqrt{6}+2\sqrt{2}} \quad \text{and} \quad \sin\frac{\pi}{24} = \frac{1}{4}\sqrt{8-2\sqrt{6}-2\sqrt{2}}$$

compute $\cos\frac{5\pi}{24}$.

Problem 2. Let $z_k, w_k \in \mathbb{C}$. Prove the Lagrange identity

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |\, z_k \,|^2\right)\left(\sum_{k=1}^{n} |\, w_k \,|^2\right) - \sum_{k<j} |\, z_k \overline{w}_j - z_j \overline{w}_k \,|^2 .$$

Then deduce the Cauchy-Schwarz inequality

$$\left|\sum_{k=1}^{n} z_k \overline{w}_k\right|^2 \leq \left(\sum_{k=1}^{n} |\, z_k \,|^2\right)\left(\sum_{k=1}^{n} |\, w_k \,|^2\right).$$

Problem 3. Prove the identity

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

if $\sin\frac{\theta}{2} \neq 0$. What is the sum if $\sin\frac{\theta}{2} = 0$? **Hint:** Show first

$$\sum_{k=-n}^{n} \mathrm{e}^{\mathrm{i}k\theta} = \frac{\cos n\theta - \cos(n+1)\theta}{1-\cos\theta} = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}.$$

The last expression is called the Dirichlet kernel or the Dirichlet-Dini kernel. It is easier to obtain it directly from the complex exponential than from the indicated intermediate expression involving cosines. It is often the case that the group property of the exponential conveniently replaces trigonometric identities. Try it both ways!

Problem 4. The first Cesaro means c_n of the partial sums of the series

$$\sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\theta}$$

are given by

$$c_n = \frac{1}{n}\sum_{k=0}^{n-1} s_k$$

where

$$s_n = \sum_{k=-n}^{n} \mathrm{e}^{\mathrm{i}k\theta} = 1 + 2\sum_{k=1}^{n} \cos k\theta.$$

Note that

$$(n+1)c_{n+1} - nc_n = s_n$$

and hence show that

$$c_n = \frac{1}{n}\frac{1-\cos n\theta}{1-\cos\theta} = \frac{1}{n}\left(\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right)^2 .$$

The last expression is called the Fejer kernel or the Cesaro-Fejer kernel.

Problem 5. Show

$$\left|\mathrm{e}^{\mathrm{i}\theta}-1\right| = 2\,\sin\frac{\theta}{2} = 2\,\Im\mathfrak{m}\left(\mathrm{e}^{\mathrm{i}\theta/2}\right).$$

Problem 6. Let $z \in \mathbb{C}$, $z \neq 0$. Let $z = x + \mathrm{i}y$ and $z^2 = u + \mathrm{i}v$ where x, y, u, v are real. Compute x and y in terms of u and v.

Problem 7. Calculate $\sqrt{3+4\mathrm{i}}$.

Problem 8. Calculate $\sqrt{4+4\mathrm{i}}$.

Problem 9. Show that $|\,z-3\mathrm{i}\,| \leq 2$ implies $|\,z+4\,| \leq 7$ and that 7 is attained.

Problem 10. The number $\xi \in \mathbb{C}$ is called an nthroot of unity if $\xi^n = 1$. The nthroots of unity form a cyclic group under multiplication. If ξ is an nthroot of unity it is said to be primitive if ξ generates the group of nthroots. Show that ξ is an nthroot of unity if and only if

$$\xi = \mathrm{e}^{\frac{2\pi\mathrm{i}k}{n}}$$

and this root is primitive if and only if k and n are relatively prime. Now show if ξ is a primitive nthroot of unity then

$$\sum_{k=1}^{n-1}\left|\xi^k-1\right| = \frac{2}{\tan\frac{\pi}{2n}}.$$