

The lectures will deal with Möbius transforms only a little bit. This note will give you some more information, and some problems to do. Turn in your solutions to any 3 problems.

We discussed the model of the Riemann sphere provided by stereographic projection in class. I will make use of it here without further comment.

If $ad - bc \neq 0$ we define the Möbius transform T by

$$T(z) = \frac{az + b}{cz + d}$$

We regard T as mapping the Riemann sphere \mathbb{C}_∞ (the one-point compactification of \mathbb{C}) into itself by defining

$$T(\infty) = \begin{cases} a/c & \text{if } c \neq 0 \\ \infty & \text{if } c = 0, \end{cases}$$

and by defining

$$T(-d/c) = \infty \text{ if } c \neq 0.$$

It follows that T is a homeomorphism of the Riemann sphere. It suffices to prove continuity since the inverse map is given by

$$T^{-1}(w) = \frac{-dw + b}{cw - a}.$$

The Möbius transforms are also called *fractional linear transforms* or (inappropriately) *bilinear transforms*. The Möbius transforms form a group \mathfrak{M} (under composition) of homeomorphisms of the Riemann sphere. The natural map

$$\Phi: GL(2, \mathbb{C}) \rightarrow \mathfrak{M}$$

is a homomorphism of groups with kernel

$$\ker(\Phi) = \{ \lambda I \mid \lambda \in \mathbb{C}, \lambda \neq 0 \}.$$

Note we could just as well regard Φ as living on the special linear group $SL(2, \mathbb{C})$ (matrices with determinant 1), in which case the kernel is just $\pm I$. An important subgroup is the *modular group* which we can view as the image of $SL(2, \mathbb{Z})$.

Other important subgroups are *translations*, $z \rightarrow z + a$ ($a \in \mathbb{C}$), *homotheties*, $z \rightarrow az$ ($a \in \mathbb{C}, a \neq 0$), *dilations* $z \rightarrow az$ ($a > 0$), *rotations* $z \rightarrow az$ ($a \in \mathbb{C}, |a| = 1$) and *linear maps* $z \rightarrow az + b$ ($a, b \in \mathbb{C}$).

Note every Möbius transform is a composition of linear maps and the inversion $z \rightarrow 1/z$ and every linear map is a composition of a homothety and a translation.

We will see later in the course that an injective analytic function has nonvanishing derivative. Thus Möbius transforms are conformal (away from $-d/c$). We can of course also see this fact by computing the derivative directly

$$T'(z) = \frac{ad - bc}{(cz + d)^2}.$$

We will write “circles” to indicate *circles and lines* in the complex plane. Under stereographic projection each circle on the Riemann sphere corresponds to a “circle” in the complex plane. Linear maps clearly take “circles” to “circles.” In addition, the inversion $z \rightarrow 1/z$, corresponds to a rotation of the Riemann sphere through an angle π about the x_1 -axis and so takes circles on the sphere to circles on the sphere. It follows that each Möbius transform maps a “circle” to a “circle” and the components of the complement (on the sphere) to the components of the complement of the image “circle.”

A quantity useful for working with Möbius transforms is the cross-ratio. Unfortunately there is no standard definition. We will define it as follows. If z_1, z_2, z_3 and z_4 are points on the Riemann sphere with z_2, z_3 and z_4 distinct then the *cross-ratio* is given by

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}.$$

Here we make the conventions that

$$\frac{a \pm \infty}{b \pm \infty} = 1, (a, b \in \mathbb{C}), \quad \frac{a}{0} = \infty, (a \neq 0), \quad \frac{a}{\infty} = 0, (a \neq \infty),$$

so, for example,

$$(z, 1, 0, \infty) = z.$$

An important property of the cross-ratio is its invariance under Möbius transforms:

Theorem 1. *If T is a Möbius transform then*

$$(T(z_1), T(z_2), T(z_3), T(z_4)) = (z_1, z_2, z_3, z_4).$$

In particular

$$T(z) = (T(z), 1, 0, \infty) = (z, T^{-1}(1), T^{-1}(0), T^{-1}(\infty))$$

Problem 1. Prove the theorem. Note it suffices to prove it for linear transforms and for the inversion.

The theorem shows the Möbius transform is determined by what it does to three points. In particular a non-identity Möbius transform can have at most two fixed points. For a translation we count ∞ as a double fixed point. For a linear map which is not a translation ∞ is a fixed point and there is a finite fixed point as well. In particular for a homothety the fixed points are 0 and ∞ . In other cases the fixed points of

$$T(z) = \frac{az + b}{cz + d}$$

are the roots (with multiplicity) of the quadratic

$$cz^2 + (d - a)z + b.$$

Note the Möbius transform T such that $T(z_j) = w_j$, $j = 1, 2, 3$ is given by

$$(T(z), T(z_1), T(z_2), T(z_3)) = (z, z_1, z_2, z_3).$$

Example 1. Let's construct a Möbius transform T mapping the real axis to the unit circle. To map the real axis to the unit circle we could for example map 1 to 1, 0 to $-i$, and ∞ to -1 . That is, we require

$$\begin{aligned} T(\infty) &= 1 \\ T(1) &= -i \\ T(0) &= -1. \end{aligned}$$

In this case

$$(T(z), 1, -i, -1) = (z, \infty, 1, 0) = \frac{z - 1}{z}$$

which yields

$$T(z) = \frac{z - i}{z + i},$$

which is the well-known Cayley transform. (You may have encountered it in Hilbert space operator theory.) Note $T(i) = 0$ so T maps the upper half plane homeomorphically to the unit disk.

Problem 2. A non-identity Möbius transform T is a homothety if and only if T has fixed points 0 and ∞ . **Hint:** Use

$$T(z) = (z, T^{-1}(1), T^{-1}(0), T^{-1}(\infty))$$

Problem 3. Let $a \neq b \in \mathbb{C}$ and let T be a nonidentity Möbius transform. Then T has fixed points a and b if and only if

$$\frac{T(z) - a}{T(z) - b} = \lambda \frac{z - a}{z - b}$$

for some nonzero complex constant λ . Moreover it then follows that

$$\frac{T^n(z) - a}{T^n(z) - b} = \lambda^n \frac{z - a}{z - b}$$

for any integer $n \geq 0$.

Note the previous exercise suggests how to define fractional powers T^t , ($t \in \mathbb{C}$), in those cases when we have a reasonable determination of $\log \lambda$:

$$\frac{T^t(z) - a}{T^t(z) - b} = \lambda^t \frac{z - a}{z - b}$$

for any integer $n \geq 0$.

Problem 4. A non-identity Möbius transform T leaves ± 1 fixed if and only if T can be written

$$T(z) = \frac{az + b}{bz + a}$$

where $a^2 \neq b^2$.

Problem 5. The non-identity Möbius transform T maps the unit circle to itself if and only if T may be written as

$$T(z) = \frac{az + b}{\bar{b}z + \bar{a}}$$

with $a \neq b$. If in addition $|b| < |a|$ then the unit disk is mapped to the unit disk.

If $|\alpha| < 1$ we define

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$$

Problem 6. $\phi_\alpha(z)$ maps the unit disk onto the unit disk.

Problem 7. If T is a Möbius transform with $T(0) = 0$ such that T maps the unit circle to itself then T is a rotation. **Hint:** If $d \neq 0$ and $ce^{it} + d$ has constant modulus then $c = 0$.

Problem 8. If T is a non-identity Möbius transform then T maps the unit disk onto the unit disk if and only if there is a real number θ and a complex number α with $|\alpha| < 1$ such that

$$T(z) = e^{i\theta} \phi_\alpha(z).$$

Hint: If $\alpha = T^{-1}(0)$ show that $S = T \circ \phi_\alpha^{-1}$ is a rotation.

Example 2. Let $a \neq b \in \mathbb{C}$ and let

$$T(z) = \frac{z - a}{z - b}.$$

If Γ_μ is a circle with center at the origin and radius μ then

$$T^{-1}(\Gamma_\mu) = \left\{ z \mid \left| \frac{z - a}{z - b} \right| = \mu \right\}.$$

These "circles" are the Circles of Apollonius. The center of $T^{-1}(\Gamma_\mu)$ is

$$c_\mu = \frac{a - \mu^2 b}{1 - \mu^2}$$

and the radius is

$$R_\mu = \frac{\mu |a - b|}{|1 - \mu^2|}.$$

Note also if we take any line L through the origin then $T^{-1}(L)$ is a circle through $a = T^{-1}(0)$ and through $b = T^{-1}(\infty)$. Since T is conformal this second set of circles, namely all the "circles" through a and b , is orthogonal to the circles of Apollonius.

Problem 9. Sketch the Circles of Apollonius and the orthogonal circles. One of the circles of Apollonius is a line - describe it.

Problem 10. Construct the Möbius transform T which satisfies $T(-i) = -i$, $T(0) = -1$ and $T(i) = i$. Describe T .

Problem 11. Let T be a Möbius transform. Then $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ if and only if T may be written as

$$T(z) = \frac{az + b}{cz + d}$$

where a, b, c and d are real. In this case T maps the upper half plane to the upper half plane if and only if $ad - bc > 0$.

Problem 12. Let $\Im \alpha > 0$. Then $T \in \mathfrak{M}$ maps the upper half plane onto the unit disk and $T(\alpha) = 0$ if and only if

$$T(z) = \lambda \frac{z - \alpha}{z - \bar{\alpha}}$$

where $|\lambda| = 1$.