100. Example. Ordinary Differential Equations 1

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In this section we show how Contraction Mapping Principle implies a version of the fundamental existence and uniqueness theorem for Cauchy initial value problems for ordinary differential equations.

This application of the Contraction Mapping Principle is also discussed in Kolmogorov and Fomin [2], chapter 2, section 8, where there is also an application to solving Fredholm integral equations of the second kind and Volterra integral equations.

Let Ω be an open subset of the plane and let f be a continuous function in Ω. Let $(a, c) \in \Omega$ and consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \;\; y(a) = c.$$

By integrating we see this initial value problem is equivalent to the integral equation

$$y(x) = c + \int_a^x f(t, y(t))\, dt,$$

that is, we are looking for a fixed point for the map defined by the integral. Let $p > 0$ and $q > 0$ be chosen so that the rectangle D defined by

$$0 \le x - a \le p, \quad |\, y - c\,| \le q$$

is contained in Ω. Let M be an upper bound for f on the rectangle D. Choose h with

$$0 < h \le \min(p, q/M).$$

Let $b = a + h$ and let

$$X_h = \{\, u \in \mathrm{C}([a, b]) \;\mid\; u(a) = c, \;\; |\, u(x) - c\,| \le q \text{ if } a \le x \le b \,\}\,.$$

If $u \in X_h$ define Tu by

$$Tu(x) = c + \int_a^x f(t, u(t))\, dt.$$

Clearly

$$|\, Tu(x) - c\,| \leq M(x-a) \leq Mh \leq q$$

and so $T\colon X_h \to X_h$. The space $\mathrm{C}([a,b])$ is a complete metric space when provided with the metric $d(u,v) = \sup |\, u(t) - v(t)\,|$, $a \leq t \leq b$ and X_h is a closed subset. Thus X_h is a complete metric space. Now suppose that f is Lipschitz continuous in the second variable in the rectangle D, say

$$|\, f(x,y) - f(x,z)\,| \leq L\ |\, y - z\,|, \qquad \text{for } (x,y), (x,z) \in D.$$

Let $0 < \delta < 1$. If $u, v \in X_h$ then

$$|\, Tu(x) - Tv(x)\,| \leq \int_a^x |\, f(t,u(t)) - f(t,v(t))\,|\ dt$$

$$\leq \int_a^x L\ |\, u(t) - v(t)\,|\ dt \leq Lh\, d(u,v).$$

Thus T is continuous. Moreover if $Lh \leq \delta$ then this inequality implies that T is a contraction mapping and so has a unique fixed point in X_h. A simple estimate shows that any solution of the initial value problem must be in X_h. Thus we have the existence of a unique solution to the initial value problem on the interval $[a,b]$ where $b = a + h$ and

$$h = \min(p, q/M, \delta/L).$$

The iterates $u_{n+1} = T(u_n)$ converge rapidly to the fixed point. This method of approximating the solution was published by Liouville as early as 1838. The general method is usually credited to Picard, 1890 (see [4]). The successive approximants are usually called *Picard iterates*.

We can do better by using the slightly strengthened Contraction Mapping Principle, which I referred to as the *Picard iteration theorem* earlier.

Let $y_0(x) = c$ and inductively define $y_{n+1} = Ty_n$. Then

$$|\, y_1(x) - y_0(x)\,| \leq \int_a^x |\, f(t, y_0(t))\,|\ dt \leq M\,(x-a)$$

and

$$|\, y_2(x) - y_1(x)\,| = |\, Ty_1(x) - Ty_0(x)\,| \leq \int_a^x L\ |\, y_1(t) - y_0(t)\,|\ dt$$

$$\leq \int_a^x LM\,(t-a)\, dt = \frac{LM(x-a)^2}{2}.$$

By induction we see

$$|\, y_n(x) - y_{n-1}(x)\,| \leq \frac{L^{n-1}M(x-a)^n}{n\,!}$$

and so

$$d(y_n, y_{n-1}) \leq \frac{M}{L}\,\frac{(Lh)^n}{n!}.$$

It follows that the sequence $(y_n)_{n\geq 0}$ converges to a fixed point of T. Thus we have existence of a solution to the initial value problem on the interval $[a, b]$ where $b = a + h$ and

$$h = \min(p, q/M).$$

Uniqueness requires an additional argument. See, for example, Nemytskii and Stepanov [3] or Coddington and Levinson [1]. Finally note that the argument is essentially unchanged if y and f are vector valued. Since any system of differential equations can be replaced by a system of first order equations, we obtain existence and uniqueness under mild hypotheses for the Cauchy initial value problem for any system of differential equations (in normal form).

References

[1] Earl A. Coddington and Norman Levinson. *Theory of Ordinary Differential Equations*. McGraw–Hill, New York · Toronto · London, 1955.

[2] A. N. Kolmogorov and S. V. Fomin. *Introductory Real Analysis*. Dover Publications, Inc., New York, 1975. Tranlated from Russian, edited and revised by Richard A. Silverman.

[3] V. V. Nemytskii and V. V. Stepanov. *Qualitative Theory of Differential Equations*, volume 22 of *Princeton Mathematical Series*. Princeton Univ. Press, Princton, New Jersey, 1960.

[4] (Charles) Emile Picard. [check title]. *Jour. de Math*, (4) 6:145–210, 1890.

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