120. Topological Spaces

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20040205

In the late 1800's set theory was developed largely by Cantor. During the same period Riemann, Ascoli, Arzela, Hadamard and others inspired by the needs of the calculus of variations introduced the concept of function spaces – sets in which the points are functions. Fréchet in his 1906 thesis [1] used abstract set theory and introduced an axiomatic approach to function spaces. He considered abstract sets provided with an abstract limit process. He also introduced abstract metric spaces and raised questions of metrizability for topological spaces. In his thesis he also applied his ideas to examples of function spaces; continuous functions on an interval, analytic functions in a domain. His thesis demonstrated that useful and interesting results could be obtained in an abstract setting. See Taylor [3], [4] and [5] for an exhaustive study of the work of Fréchet and its relation to the work of his contemporaries.

In this section we will assume some familiarity with topology and therefore merely review some of the basic ideas. This review serves to fix terminology. The notions of sequential convergence and continuity may be phrased in terms of neighborhoods and open sets. One is therefore led to the idea of abstracting the concept of an open set or a neighborhood. The resulting topic is general topology.

- *If we take neighborhoods to be our primitive concept then we define the open sets to be those sets which are neighborhoods of each of their points.*
- *If we take open sets to be the primitive concept then we declare V to be a neighborhood of $x \in X$ if there exists an open set U with $x \in U \subseteq V$.*

Topology may be introduced in other ways as well by using different primitive concepts but open sets or neighborhoods are the most popular. We will take the point of view that a topology is defined by its open sets, though we will frequently describe a topology by specifying its neighborhoods.

If X is a set, the *power set* of X, frequently denoted by 2^X, is the set of all subsets of X. We define a *topology* on the set X to be a nonempty subset τ of 2^X satisfying the axioms that $\varnothing$ and X in τ and finite intersections and arbitrary unions of elements of τ are in τ. The elements of τ are called the open sets.

Exercise 120.1. *Assume we have a topology τ as above on a set X. If $x \in X$ we define U to be a neighborhood of x if there exists an open set V, that is, a set $V \in \tau$, such that $x \in V \subseteq U$. Show that a subset W of X is in τ if and only if for each $x \in W$ we have W is a neighborhood of x.*

- The *interior* A° of a set A is its largest open subset. This definition makes sense since the union of any number of open sets is open.
- If $A \subseteq X$ then $V \subseteq X$ is a *neighborhood* of A if $A \subseteq V^\circ$.
- The i *closure* $\overline{A}$ of a set A is its smallest closed superset. This definition makes sense since the intersection of any number of closed sets is closed.
- The *exterior* of a set is the interior of its complement.
- The *boundary* ∂A of A is the complement of the union of the interior of A and the exterior of A.
- If A and B are sets we denote by $A \sim B$ the complement of B in A, that is, $\{\, x \in A \mid x \notin B \,\}$.
- If τ is a topology on X and $A \subseteq X$ then $\tau_A = \{\, U \cap A \mid U \in \tau \,\}$ is a topology on A called the *relative topology* or the *induced topology*.

Let $A \subseteq X$ and provide A with the relative topology. Let $y \in V \subseteq A$. Then V is a relative neighborhood of y if and only if there is a neighborhhod W of y in X such that $A \cap W = V$. Indeed if W is a neighborhood of y in X and $A \cap W = V$ choose an open set U in X with $y \in U \subseteq W$. Then $y \in A \cap U \subseteq V$ implies V is a relative neighborhood of y since $A \cap U$ is relatively open by definition. Conversely let V be a relative neighborhood of y in A. By definition of the relatively open sets we can choose U open in X with $U \cap A \subseteq V$. Now let $W = U \cup V$. Since $y \in U$ and U is open we see W is a neighborhood of y. But

$$W \cap A = (U \cap A) \bigcup (V \cap A) = (U \cap A) \bigcup V = V$$

since $U \cap A \subseteq V$.

Exercise 120.2. *Let X be a set and let $\mathfrak{S}$ be any family of subsets of X. Show there exists a unique weakest topology on X containing $\mathfrak{S}$, that is, for which each set in $\mathfrak{S}$ is open.* **Hint:** *The intersection of any number of topologies on X is a topology on X.*

Exercise 120.3. *Show $\partial A = \overline{A} \cap \overline{X \sim A}$, that is, $x \in \partial A$ if and only if each neighborhood of x intersects A and the complement of A. Show if U is open then $\partial U = \overline{U} \sim U$.*

Topologies are partially ordered by inclusion. If σ and τ are topologies (families of open sets) on a set X we say that σ is *weaker* than τ or τ is *stronger* than σ if $\sigma \subseteq \tau$. The weakest topology on X is called the *trivial topology* on X. The strongest topology on X is called the *discrete topology* on X.

In topological vector spaces the most convenient way to prescribe a topology is by means of its neighborhoods. Thus it is necessary to also consider topology from the point of view of abstract axiomatic neighborhood systems.

Let X be a set. An *abstract neighborhood system* on X consists of a nonempty subset $\mathfrak{N}_x$ of 2^X for each $x \in X$ such that

1. $A \in \mathfrak{N}_x$ implies $x \in A$
2. $A \in \mathfrak{N}_x$ and $A \subseteq B$ implies $B \in \mathfrak{N}_x$
3. $A,\, B \in \mathfrak{N}_x$ implies $A \cap B \in \mathfrak{N}_x$
4. If $A \in \mathfrak{N}_x$ there exists $U \in \mathfrak{N}_x$ such that $U \subseteq A$ and $U \in \mathfrak{N}_y$ for each $y \in U$

The elements of $\mathfrak{N}_x$ are called the *neighborhoods* of x. We paraphrase the last condition by saying each neighborhood is a neighorhood of "nearby" points. If we define open sets to be the sets that are neighborhoods of each of their points then the first three axioms for an abstract neighborhood system suffice to guarantee that our open sets satisfy the axioms for a topology. The last axiom is needed to guarantee that if we now define neighborhoods in terms of the open sets we get back exactly the neighborhoods that we started with.

Exercise 120.4. *Let X be a set. Describe the open sets, the neighborhoods and the closures of subsets of X for the trivial and for the discrete topologies on X.*

It's usually not convenient to describe all of the neighborhoods of a topology. Instead one describes an neighborhood base or subbase.

A set $\mathfrak{B}_x \subseteq \mathfrak{N}_x$ is called a *neighborhood base* at x if for each $V \in \mathfrak{N}_x$ there is $W \in \mathfrak{B}_x$ such that $W \subseteq V$.

A set $\mathfrak{B}_x \subseteq \mathfrak{N}_x$ is called a *neighborhood subbase* at x if for each $V \in \mathfrak{N}_x$ there is a finite set $W_1, \ldots W_n \in \mathfrak{B}_x$ such that $W_1 \cap \cdots \cap W_n \subseteq V$.

One can of course axiomatize the concepts of neighborhood base or neighborhood subbase and introduce topology that way. This idea is actually a good one since in practice many topologies are introduced by specifying a neighborhood base at each point. In a semimetric space X the sets $W(x, \epsilon)$, $\epsilon > 0$, form a base of neighborhoods of x, as do the sets $B(x, \epsilon)$, $\epsilon > 0$.

Exercise 120.5. *You are writing a topology text and decide to use neighborhood bases (or even subbases) as the primitive concept to be axiomatized. How would you do it? Be sure to introduce a suitable (primitive) notion of equivalence for two neighborhood bases (subbases).*

If X is a topological space and A is a subset of X then $V \subseteq X$ is called a *neighborhood* of A if $A \subseteq V^\circ$. If X is a metric space then the *open* ϵ*–neighborhood* is defined to be

$$W(A, \epsilon) = \{\, y \in X \ \mid\ d(x, y) < \epsilon \text{ for some } x \in A \,\} = \bigcup_{x \in A} W(x, \epsilon)$$

is an open neighborhood of the set A.

There are a number of *separation axioms* in topology. Since there is no universal agreement on how several of them should be formulated let's review the main ones here.

Let X be a topological space. Then X is

- T_0 (Kolmogorov) if whenever $a \neq b$ are points of X there exists an open subset U of X such that $a \in U$ and $b \notin U$ or $a \notin U$ and $b \in U$.
- T_1 (Fréchet–Riesz) if whenever $a \neq b$ are points of X there exist open subsets U, V of X such that $a \in U$, $b \in V$, $a \notin V$ and $b \notin U$.
- T_2 (Hausdorff) if whenever $a \neq b$ are points of X there exist disjoint open sets U, V with $a \in U$ and $b \in V$.
- T_3 (Vietoris) if for each $a \in X$ and each closed subset A of X with $a \notin A$ there exist disjoint open subsets U, V of X such that $a \in U$ and $A \subseteq V$.
- *regular* if T_3 and T_0.
- T_4 (Tietze) if whenever A, B are disjoint closed subsets of X there exist disjoint open subsets U, V of X with $A \subseteq U$ and $B \subseteq V$.
- *normal* if T_4 and T_1.
- T_5 (Urysohn) if whenever A, B are subsets of X and $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$ there exist disjoint open subsets U, V of X with $A \subseteq U$ and $B \subseteq V$.
- *completely normal* if T_5 and T_1.

A semimetric space is T_5. A metric space is completely normal. A topological space is completely normal if and only if each subspace is normal. The next three lemmas are key properties of T_4 spaces. The first two are particularly important for analysis because they show that T_4 spaces are richly endowed with continuous functions. Proofs may be found in Gaal [2] and other general topology texts.

Lemma 120.1 (Urysohn Lemma). *Let X be a topological space. Then X is a T_4 space if and only if for each pair of disjoint closed subsets A, B of X there exists a continuous function $f : X \to [0, 1]$ such that $f_{|A} = 1$ and $f_{|B} = 0$.*

Lemma 120.2 (Tietze Extension Lemma). *Let X be a topological space. Then X is a T_4 space if and only if for each closed subset A of X and each continuous function $f: A \to \mathbb{R}$ has a continuous extension to X. Moreover, in the case X is T_4 and f is bounded the extension may be chosen to be bounded and to have the same bound as f.*

Let X be a set and let A be a subset of X. If $(U_i)_{i\in I}$ is a family of subsets of X we say this family is a *cover* of A if $A \subseteq \cup_{i\in I} U_i$. The indexed family $(U_i)_{i\in I}$ of subsets of X is said to be *point finite* if for each $x \in X$ we have $\{\, i \in I \;\mid\; x \in U_i \,\}$ is finite. In case X is a topological space the cover $(U_i)_{i\in I}$ is said to be an *open cover* if each of the sets U_i is open. The indexed family $(U_i)_{i\in I}$ of subsets of X is said to be *locally finite* if for each $x \in X$ we have a neighborhood V of x such that $\{\, i \in I \;\mid\; V \cap U_i \neq \varnothing \,\}$ is finite.

Lemma 120.3 (Shrinking Lemma). *If $(U_i)_{i\in I}$ is a point finite open cover of a T_4 space X then there exists an open cover $(V_i)_{i\in I}$ such that $\overline{V_i} \subseteq U_i$ for each $i \in I$.*

The notion of continuity of a function at a point has a nice formulation in terms of neighborhoods: f is continuous at a if and only if for each neighborhood V of $f(a)$ there is a neighborhood W of a such that $f(W) \subseteq V$.

The notion of continuity also has a nice formulation in terms of open sets: f is continuous if and only if for each open set $V \subseteq Y$ the set $f^{-1}(V)$ is an open subset of X.

These definitions agree with the definitions of continuity of functions in metric spaces.

The notion of *uniform continuity* does not make sense in topological spaces. The correct context is *uniform space*. Topological groups, topological vector spaces, and metric spaces are all uniform spaces in a natural way and so uniform continuity makes sense in those contexts. Here is the definition for metric spaces: If X and Y are metric spaces (with the metric denoted by d in both cases) then $f: X \to Y$ is uniformly continuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that $a, b \in X$, $d(a,b) < \delta$ implies $d\,(f(a), f(b)) < \epsilon$. As you can see, the definition requires some way of associating neighborhoods at different locations (in a metric space, vaguely speaking, by size). It is precisely this facility that the *vicinities* of the uniform structure provide - each vicinity gives rise to a distinguished neighborhood of each point.

References

[1] Maurice Fréchet. Sur quelques points du calcul fonctionnel. *Rendiconti del Circolo Matematico di Palermo*, 22:1–74, 1906.

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[3] Angus E. Taylor. A study of Maurice Fréchet: I His early work on point set theory and the theory of functionals. *Archive for History of Exact Sciences*, 27:233–295, 1982.

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[5] Angus E. Taylor. A study of Maurice Fréchet: III Fréchet as analyst, 1909–1930. *Archive for History of Exact Sciences*, 37:25–76, 1987.

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