140. Baire Category Theorem

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Baire's theorem was discovered in the late 1890's by Baire [1] and also by Osgood [4]. In the late 1920's Baire's theorem was introduced into Functional Analysis by Banach and Steinhaus.

Where did this comment about Banach and Steinhaus come from? I need a citation.

Let X be a topological space and let B be a subset of X. We say B is *dense in* X if $\overline{B} = X$, that is, if for each nonempty open subset U of X we have $B \cap U \neq \varnothing$. Thus B is dense in X if and only if the exterior of B is empty.

Exercise 140.1. *A finite intersection of open dense subsets of the topological space X is an open dense set.*

Let X be a topological space and let A be a subset. We say A is *nowhere dense* if $\left(\overline{A}\right)^\circ = \varnothing$, that is, if the complement $X \sim A$ contains an open subset of X dense in X.

Exercise 140.2. *A finite union of sets nowhere dense in the topological space X is nowhere dense in X.*

We say $A \subseteq X$ is of *first category in* X or *meagre in* X if A is a countable union of nowhere dense subsets of X, that is, if the complement $X \sim A$ contains a countable intersection of open dense subsets of X. Clearly any countable union of first category sets is of first category.

We say $A \subseteq X$ is of *second category in* X or *nonmeagre in* X if A is not of first category in X.

A *residual set* in X is a set which is the complement of a set of first category in X.

Exercise 140.3. *A subset A of X is second category (nonmeagre) in X if and only if whenever $(V_n)_{n\geq 1}$ is a sequence of open dense subsets of X then*

$$A \cap \bigcap_{n\geq 1} V_n \neq \varnothing.$$

A topological space X is called a *Baire space* if any countable intersection of open dense subsets of X is dense in X. Clearly we have

- *X is Baire if and only if each residual subset of X is dense in X.*
- *X is a Baire space if and only if each nonempty open subset of X is second category in X.*
- *X is a Baire space if and only if each first category subset of X has empty interior.*

Proposition 140.1. *Let X be a topological space and let U be an open subset of X. If $A \subseteq U$ and A is nowhere dense in U in the relative topology, then A is nowhere dense in X.*

Proof. Since A is nowhere dense in U there is an open set $W_0 \subseteq U \sim A$ such that W_0 is dense in U. Let W_1 be the interior of $W_0 \cup (X \sim U)$. Let V be any nonempty open subset of X. If $V \cap U = \varnothing$ then $V \subset X \sim U$ and so $V \subset W_1$ since V is open. On the other hand, if $V \cap U \neq \varnothing$ then this set is a nonempty open subset of U and therefore meets W_0. In either case we have $V \cap W_1 \neq \varnothing$ and so W_1 is dense in X. Since W_1 is open and contained in $X \sim A$ it follows that A is nowhere dense in X. □

Exercise 140.4. *If $\mathfrak{U}$ is a family of disjoint open subsets of the topological space X and for each $U \in \mathfrak{U}$ we have a set $A_U \subseteq U$ such that A_U is nowhere dense in X then*

$$A = \bigcup_{U\in\mathfrak{U}} A_U$$

is nowhere dense in X.

Corollary 140.2. *Let X be a topological space and let U be an open subset of X. If $A \subseteq U$ and A is first category in U then A is first category in X.*

Corollary 140.3. *Let X be a topological space and let U be an open subset of X. If $A \subseteq U$ and A is second category in X then A is second category in U.*

Corollary 140.4. *Each open subset of a Baire space is a Baire space.*

Theorem 140.5. *Let X be a topological space and let A be a subset of X. Let C be the union of all open subsets U of X suct that $A \cap U$ is of first category in X. Then $A \cap \overline{C}$ is of first category in X.*

Proof. See Kelley, [3]. □

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Let X be a topological space and let A be a subset of X. We say A is *locally of first category* in X or *locally meager* in X if each point $x \in X$ has an open neighborhood V such that $A \cap V$ is first category in X.

Corollary 140.6. *Let X be a topological space and let A be a subset of X. If A is locally of first category in X the A is of first category in X.*

Proof. Indeed the hypotheses imply $A \subseteq C$ (in the notation of the theorem). □

Corollary 140.7. *Let X be a topological space and let A be a subset of X. If A is second category in X then there exists a nonempty open subset V of X such that if $x \in \overline{V}$ then $A \cap U$ is of second category for each neighborhood U of x.*

Proof. In the notation of the theorem, by hypothesis $\overline{C} \neq X$. Let V be the complement of $\overline{V}$. If $x \in \overline{V}$ and U is an open neighborhood of x then $U \cap V \neq \varnothing$ (definition of closure). Thus U is not contained in C and so $A \cap U$ is not of first category. □

Theorem 140.8 (Baire Category Theorem). *Let X be either a complete semimetric space or a locally compact Hausdorff space. Then X is a Baire space.*

Proof. We give the proof only in the semimetric case. The proof in the locally compact T_2 case is remarkly similar. Let (V_n) be a sequence of open dense subsets of X and let U be any nonempty open subset of X. We must show that $\cap_{n\geq 1} V_n \cap U \neq \varnothing$. Choose $x_0 \in U$ and $r_0 > 0$ such that $W(x_0, r_0) \subseteq U$. We will inductively choose $x_n \in U$ and $r_n > 0$ for $n \geq 1$. Let $n \geq 1$. Since V_n is open and dense the intersection $V_n \cap W(x_{n-1}, r_{n-1})$ is a nonempty open set. Hence we may choose r_n with $0 < r_n < 2^{-n}$ and $x_n \in U$ such that

$$B(x_n, r_n) \subseteq V_n \cap W(x_{n-1}, r_{n-1}).$$

Since $x_n \in W(x_{n-1}, r_{n-1})$ we have $d(x_{n-1}, x_n) < 2^{1-n}$ for $n > 1$. Thus (x_n) is a Cauchy sequence. Since X is complete this sequence converges, say $x_n \to x$. Since $x_k \in B(x_n, r_n)$ for $k \geq n$ we have $x \in B(x_n, r_n)$ for each $n \geq 1$. Thus $x \in V_n$ for each $n \geq 1$ and $x \in B(x_1, r_1) \subseteq W(x_0, r_0) \subseteq U$. □

Proposition 140.9. *Let X be a Baire space and let $(A_n)_{n\geq 1}$ be a sequence of closed subsets of X. If $X = \cup_n A_n$ then*

$$\bigcup_{n=1}^{\infty} A_n^{\circ}$$

is dense in X.

Proof. Suppose first that $A_n^\circ = \varnothing$ for each n. Then $U_n = X \sim A_n$ is a dense open set and so $\varnothing = \cap_n U_n$ is dense in X. Since we may obviously assume that X is not empty we have a contradiction. Thus we have proved

$$\bigcup_{n=1}^{\infty} A_n^\circ \neq \varnothing.$$

Now let U be any nonempty open subset of X. Then $U \cap A_n$ is a relatively closed subset of U, $U = \cup_n U \cap A_n$ and U is a Baire space. Hence by the part proved above we have

$$U \cap \bigcup_{n=1}^{\infty} A_n^\circ \neq \varnothing.$$

Thus $\cup_n A_n^\circ$ is dense in X. □

Exercise 140.5. *A topological space X is a Baire space if and only if each subset of first category has dense complement in X.*

Let l^p be the space of p–summable sequences with the usual metric, so l^p is a complete metric space. (We will discuss l^p later.)

Exercise 140.6. *If $(a^n)_{n\geq 1}$ is a sequence in l^1, say $a^n = (a_1^n, a_2^n, \cdots)$ and we define $a_j = \liminf_{n\to\infty} \left|a_j^n\right|$ show that $\|a\|_1 \leq \liminf_{n\to\infty} \|a^n\|_1$.*

Exercise 140.7. *Let B be the unit ball in l^1. Show that B is closed in l^2. Show B has empty interior in l^2 and hence l^1 is first category in l^2. Conclude*

$$\left\{ b \in l^2 \;\mid\; \sum_j |b_j| = \infty \right\}$$

is dense in l^2.

Example 140.1. **(First category sets of positive measure.)** This example is adapted from examples 4, 19 and 20 in section 8 of Gelbaum and Olmsted [2] (Every analyst should own a copy of this book). While sets of first category may be thought of as somehow small or thin in the sense of Baire, they can be quite *large* in, for example, the sense of measure. Here's a traditional Cantor-like construction. Let $0 < t < 1$ and let $B_0 = [0, 1]$. Suppose we have constructed $B_k,\ 0 \leq k < n$ and each B_k consists of 2^k disjoint uniformly distributed closed intervals of lengths $2^{-k}\left(1 - 2^{-1}t - \cdots - 2^{-k}t\right) = 2^{-k}\left(1 - t(1 - 2^{-k})\right) > t2^{-2k}$. Each of the closed intervals making up B_{n-1} has length $> t2^{2-2n}$ and therefore we can remove from each the middle open interval of length $t2^{1-2n} < t2^{2-2n}$. This construction yields B_n. Now let

$$A(t) = \bigcap_{n=1}^{\infty} B_n.$$

The longest interval in B_n has length $2^{-n}\left(1 - 2^{-1}t - \cdots - 2^{-n}t\right) < 2^{-n}$. Hence $A(t)$ contains no nonempty intervals and therefore $A(t)^\circ = \varnothing$. Since $B_n \downarrow A(t)$ and B_n has Lebesgue measure $2^n \times 2^{-n}\left(1 - 2^{-1}t - \cdots - 2^{-n}t\right)$ we see that $A(t)$ has measure $1 - t$. Since $A(t)$ is closed we have

Fact . $A(t)$ *is a closed nowhere dense subset of* $[0, 1]$ *with Lebesgue measure* $1 - t$.

Now let $A_n = A(1/n)$ and let $A = \cup_{n \geq 1} A(1/n)$. Then

Fact . A *is a first category subset of* $[0, 1]$ *and* A *has Lebesgue measure* 1.

Consider now the complement $C = [0, 1] \sim A$. Since $[0, 1]$ is a complete metric space, $[0, 1] = A \cup C$ and A is first category we see by Baire's theorem that C cannot be first category. Thus

Fact . C *is a dense second category subset of* $[0, 1]$ *and* C *has Lebesgue measure* 0.

References

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[2] Bernard R. Gelbaum and John M. H. Olmsted. *Counterexamples in Analysis*. Holden–Day, San Francisco, 1964.

[3] John L. Kelley. *General Topology*. van Nostrand, Princeton, New Jersey, 1955.

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