160. Compactness

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Fréchet in his thesis in 1906 introduced and studied notions related to compactness, separability and completeness and discussed the relation between compactness and total boundedness, all in abstract spaces.

The assertion that a certain set A is compact is an existence theorem. See for example Exercise 160.3 below. Or recall a continuous real valued function on a compact space has a maximum (and a minimum). For this reason, the identification of compact subsets is important, especially compact subsets of function spaces.

Let X be a topological space and let A be a subset. We say that A is *compact* if each open cover of A in X has a finite subcover (Heine–Borel property) . It is not difficult to see that A is compact if and only if A is compact in the relative topology on A. Thus compactness is an intrinsic property. A subset A of X is said to be *relatively compact* if it is contained in a compact subset of X.

Every closed subset of a compact set is compact and the intersection of any number of closed compact sets is compact. The continuous image of a compact set is compact. Compactness is not too well behaved in non–Hausdorff spaces. Thus the closure of a compact set may fail to be compact and the intersection of two compact sets may fail to be compact.

Exercise 160.1. *If X is a Hausdorff topological space then each compact subset of X is closed and a subset A of X is relatively compact if and only if its closure is compact.*

Exercise 160.2. *Let X be a compact topological space and let U be an open subset of X. Let $(K_j)_{j\in J}$ be a family of compact subsets of X. If $\cap_{j\in J} K_j \subseteq U$ then there is a finite set $F \subseteq J$ such that $\cap_{j\in F} K_j \subseteq U$.*

Let X be a set. A family of subsets $\mathfrak{C}$ of X is said to have the *finite intersection property* if each finite subfamily has non-empty intersection. Cantor is usually credited as having formulated the connection between compactness and the finite intersection property as indicated in the next exercise.

How about a citation?

Exercise 160.3. *Let X be a topological space. Then X is compact if and only if each family of closed subsets of X with the finite intersection property has nonempty intersection.*

Exercise 160.4. *Let X be a compact topological space and let $(K_j)_{j \in J}$ be a family of nonempty closed subsets of X with $K_{j+1} \subset K_j$ for each $j \geq 1$. The $\cap_{j \geq 1} K_j \neq \varnothing$.*

In a metric space a set A is compact if and only if each sequence in A has a subsequence converging to a point of A (Bolzano–Weierstrass property) . This latter property is referred to as *sequential compactness*. The open covering approach to compactness is due to Heine, Borel, Cousin and Lebesgue and is the currently accepted definition. These two notions of compactness are not equivalent in a general topological spaces.

Exercise 160.5. *Let X be a compact metric space, A a closed subset of X and V an open subset of X. Suppose $A \subseteq V$. Show there exists $\epsilon > 0$ such that*

$$\bigcup_{x \in A} W(x, \epsilon) \subseteq V.$$

The following simple fact is remarkably useful.

Lemma 160.1 (Uniqueness of Compact Topology). *Let X and Y be topological spaces. Assume X is compact and Y is Hausdorff. If $f : X \to Y$ is a continuous bijection then f is a homeomorphism.*

Proof. It suffices to show that f^{-1} is continuous, for which it suffices to show that f is closed. Let A be a closed subset of X. By hypothesis A is compact. Thus $f(A)$ is compact and so closed since Y is Hausdorff. ☐

A topological space X is said to be *locally compact* if each point in X has a compact neighborhood. Compact and locally compact spaces have some nice separation properties.

Lemma 160.2. *A compact* T_2 *space is* T_4.

Proof. Suppose X is compact and T_2. Let A and B be disjoint closed subsets of X. If $a \in A$, $b \in B$ we may choose disjoint open subsets $U_{a,b}$ and $V_{a,b}$ of X with $a \in U_{a,b}$ and $b \in V_{a,b}$. Let $a \in A$. The sets $V_{a,b}$ form an open cover of B. By compactness there is a finite set $M_a \subseteq B$ such that $B \subseteq \cup_{b \in M_a} V_{a,b} = V'_a$. Then U'_a and V'_a are disjoint open sets, $a \in U'_a$ and $B \subseteq V'_a$. Hence there is a finite set $M \subseteq A$ such that $A \subseteq \cup_{a \in M} U'_a = U$. Now let $V = \cap_{a \in M} V'_a$. Then $A \subseteq U$, $B \subseteq V$ and U and V are disjoint and open. ☐

Lemma 160.3. *A* T_1 *space is* T_3 *if and only if each point has a neighborhood base consisting of closures of open neighborhoods.*

Proposition 160.4. *If X is a locally compact space and X is* T_2 *or* T_3 *then each point x has a neighborhood base consisting of sets of the form $\overline{V}$ where V is an open neighborhood of x and $\overline{V}$ is compact.*

Proof. Suppose first that X is T_3. Let U be any neighborhood of x. Let W be a compact neighborhood of x. By the lemma above there is an open neighborhood V of x with $\overline{V} \subseteq U \cap W$. Now consider the case X is T_2. Let $x \in X$ and let U be any neighborhood of x. Again select a compact neighborhhod W of x. Let $V = (U \cap W))^\circ$ so V is an open neighborhood of x. Since X is T_2 W is closed and therefore $\overline{V} \subseteq W$. Thus $\overline{V}$ is compact. Since V is open and $x \in V$ we have $\{x\}$ and ∂V are disjoint. Since $\overline{V}$ is compact there are disjoint relatively open set U_1 and V_1 in $\overline{V}$ such that $x \in U_1$ and $\partial V \subseteq V_1$. Since $U_1 \subseteq \overline{V}$ is disjoint from ∂V we have $U_1 \subseteq V$. Since V is open it follows that U_1 is open in X. Since V_1 is relatively open in $\overline{V}$ we have $\overline{U_1} \subseteq \overline{V} \sim V_1 \subseteq V$ since $\partial V \subseteq V_1$. Thus $\overline{U_1} \subseteq U$, $\overline{U_1}$ is compact and U_1 is open. ☐

If $f : X \to \mathbb{C}$ is continuous we define the *support* of f as the closure of $\{\, x \in X \mid f(x) \neq 0 \,\}$ and we denote it by $\operatorname{supp} f$. Here $\mathbb{C}$ denotes the field of complex numbers with the usual metric. We will define the concept of support in other contexts as well but the present definition suffices for now.

Lemma 160.5. *If X is a locally compact* T_2 *space, U is open in X, K is compact in X and $K \subseteq U$ then there exists a continuous function $f : X \to [0,1]$ such that $\operatorname{supp} f \subseteq U$ and $f = 1$ in a neighborhood of K.*

Proof. For each $x \in K$ choose a compact neighborhood V_x of x with $V_x \subseteq U$. Choose a finite set $N \subseteq K$ such that the sets V_x° with $x \in N$ cover K and let $A = \cup_{x \in N} v_x$. Then A is compact and $K \subseteq A^\circ$. Since A is compact we can repeat the argument and end up with compact sets A, B, C such that

$$K \subseteq A^\circ \subseteq A \subseteq B^\circ \subseteq B \subseteq C^\circ \subseteq C \subseteq U.$$

Since C is compact and T_2 it is T_4. Thus by Urysohn's lemma there is a continuous function $f : C \to [0,1]$ such that $f = 1$ on A and $f = 0$ on $C \sim B^\circ$. Set $f = 0$ on $X \sim C$. If $x \in C^\circ$ or $x \in X \sim C$ then f is continuous at x. Suppose $x \in \partial C$. In this case $f = 0$ on the relatively open set $C \sim B \subseteq C \sim B^\circ$. Choose an open set V with $x \in V$ and $V \cap C \subseteq C \sim B$. Then $f = 0$ on $V \cap C$ and by construction $f = 0$ on $V \cap (X \sim C)$. Thus $f = 0$ on V and so f is continuous at x. Finally $\{\, x \in X \mid f(x) \neq 0 \,\} \subseteq B^\circ$ implies that $\operatorname{supp} f \subseteq B \subseteq U$. ☐

Theorem 160.6 (Finite Partition of Unity). *Let X be a locally compact* T_2 *space and let K be a compact subset of X. If $U_1, \ldots, U_n$ are open subsets of X and $K \subseteq \cup_{1 \le j \le n} U_j$ then there exist continuous functions $h_j : X \to [0,1]$ such that $\operatorname{supp} h_j$ is compact and $\operatorname{supp} h_j \subseteq U_j$ for each j and such that $h_1 + \cdots + h_n = 1$ in a neighborhood of K.*

Proof. If $x \in K$ choose an open neighborhood V_x of x with $\overline{V_x}$ compact such that $V_x \subseteq U_j$ for some j. Then choose a finite set $N \subseteq K$ such that $K \subseteq \cup_{x \in N} V_x$. Let H_j be the union of $\overline{V_x}$ for $x \in N$ and $\overline{V_x} \subseteq U_j$. Then H_j is compact, $H_j \subseteq U_j$ and $K \subseteq H_1 \cup \cdots \cup H_n$. Now by the lemma above we can choose $g_j : X \to [0, 1]$ continuous such that $g_j = 1$ on a neighborhood of H_j, $\operatorname{supp} g_j$ is compact and $\operatorname{supp} g_j \subseteq U_j$ for each j. Now define $h_1 = g_1$, $h_2 = (1 - g_1)g_2$, $h_3 = (1-g_1)(1-g_2)g_3, \ldots, h_n = (1-g_1)(1-g_2)\ldots(1-g_{n-1})g_n$. Then $\operatorname{supp} h_j \subseteq \operatorname{supp} g_j$ is compact and contained in U_j and $h_1 + \cdots + h_n = 1 - (1 - g_1)(1 - g_2)\ldots(1 - g_n) = 1$ in a neighborhood of K. ☐

A compactness result critical in the study of weak topologies in topological vector spaces is the Tychonoff product theorem. Suppose we have a family of topological spaces, X_i, $i \in I$. We define the product $\prod_{i\in I} X_i$ to be the set of all functions $f : I \to \cup_{i\in I} X_i$ such that $f(i) \in X_i$ for each $i \in I$. Now if U_i is open in X_i and $U_i = X_i$ for all but finitely many $i \in I$ then we define $\prod_{i\in I} U_i$ to be an *open rectangle* in $\prod_{i\in I} X_i$. The *product topology* is then the topology whose family of open sets in $\prod_{i\in I} X_i$ contains all the open rectangles and is as small as possible.

Theorem 160.7 (Tychonoff Theorem). *If X_i, $i \in I$ is any family of compact topological spaces then the product space $\prod_{i\in I} X_i$ is compact.*

For a proof of the Tychonoff theorem see for example [1], page 25.

One other topological theorem that we will make some use of is a famous theorem of Hahn-Mazurkiewicz which characterizes continuous images of $[0, 1]$ in Hausdorff topological spaces. In particular it shows that there are some remarkable space filling curves.

A topological space X is said to be *connected* if it can not be written as the union of two nonempty disjoint open subsets. It is *locally connected* if each point has a base of neighborhoods consisting of connected neighborhoods.

A compact connected locally connected topological space is called a *Peano continuum*. A *Peano curve* is any space of the form $f([0, 1])$ where $f : [0, 1] \to X$ is a continuous map and X is a Hausdorff topological space.

Theorem 160.8 (Hahn–Mazurkiewicz Theorem). *A topological space Y is a Peano curve if and only if it is a metrizable Peano continuum.*

The Hahn–Mazurkiewicz theorem is discussed in the topology text of Hocking and Young [1].

References

[1] John G. Hocking and Gail S. Young. *Topology*. Addison–Wesley Publ. Co., Reading, Massachusetts · London, 1961.

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