40. Metric Spaces

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The theory of abstract metric spaces was largely created by Fréchet and Hausdorff. Indeed the name *metric space* (metrischer Raum) seems to be due to Hausdorff. The familiar neighborhood formulation of topology is due largely to Hilbert, F. Riesz, Fréchet and especially Hausdorff. Hausdorff's 1914 book [1] became a widely used standard text.

An inexpensive and useful text with a chapter on metric spaces is Kolmogorov and Fomin, [2].

In later sections we will discuss three fundamental existence theorems in the theory of complete metric spaces: the Banach Contraction Mapping Principle, the Baire Category Theorem, and the characterization of compact sets in terms of total boundedness.

Let X be a set. A *semimetric* (also called a *quasimetric*) on X is a function

$$d: X \times X \to \mathbb{R}$$

such that

1. $d(x, y) \geq 0, \ \ d(x, x) = 0$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

for each $x,\ y,\ z \in X$. If in addition we require $d(x, y) = 0 \ \Rightarrow \ x = y$ then d is called a *metric*. A *semimetric space* (respectively, a *metric space*) is a set X together with a semimetric (respectively, a metric) d on X. Most of the results obtained for metric spaces apply equally well to semimetric spaces, though some care is needed, as, for example, a compact set in a semimetric space need not be closed, limits of convergent sequences need not be unique, and so forth.

The most familiar example of a metric space is the set of real numbers $\mathbb{R}$ with the usual metric

$$d_1(x, y) = |x - y|, \quad x,\, y \in \mathbb{R}.$$

Another useful metric on $\mathbb{R}$ is given by

$$d_2(x, y) = |\arctan(x) - \arctan(y)|.$$

Given $x \in X$ and $\epsilon > 0$ we define the *open ϵ–ball at x* to be

$$W(x, \epsilon) = \{\, y \in X \;\mid\; d(x, y) < \epsilon \,\}\,.$$

A subset U of X is said to *open* if for each $x \in U$ there is $\epsilon > 0$ such that $W(x, \epsilon) \subseteq U$. Note that the empty set $\varnothing$ and the whole space X are open.

A subset V of X is said to be a *neighborhood* of x if $W(x, \epsilon) \subseteq V$ for some $\epsilon > 0$.

A subset A of X is *closed* if its complement is open. An example of a closed set is the *closed ϵ–ball at x*

$$B(x, \epsilon) = \{\, y \in X \;\mid\; d(x, y) \le \epsilon \,\}\,.$$

The *closure* of a subset A of X is the intersection of all closed supersets of A. It is easy to see that the closure of A is closed.

The *interior* of a subset A of X is the union of all open subsets of A. It is easy to see that the interior of A is open.

Exercise 40.1. *Show that $B(x, \epsilon)$ is closed. Give an example where the closure of $W(x, \epsilon)$ is a proper subset of $B(x, \epsilon)$. Show that $W(x, \epsilon)$ is open.*

Two important notions introduced early in the study of metric spaces are convergence of sequences and continuity of functions. Initially these concepts are given a metric formulation, but they are usually quickly rephrased in terms of neighborhoods and open sets.

Let $(x_n)_{n\ge1}$ be a sequence in the metric space X. We say that $(x_n)_{n\ge1}$ *converges* to $x \in X$ and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x$$

if for each $\epsilon > 0$ there is N such that

$$d(x, x_n) < \epsilon \text{ if } n \ge N.$$

We can formulate the notion of convergence entirely in terms of neighborhoods with no direct reference to the metric as follows: $x_n \to x$ if and only if for each neighborhood V of x there exists N such that $x_n \in V$ for each $n \ge N$.

Let X and Y be metric spaces and let $f : X \to Y$. Let $a \in X$. We say that f is *continuous at* a if for each $\epsilon > 0$ there is $\delta > 0$ such that $x \in X$, $d(x, a) < \delta$ implies $d(f(x), f(a)) < \epsilon$. We say f is *continuous on* X or simply *continuous* if f is continuous at each point of X.

Another important notion in metric space is uniform continuity. Let X and Y be metric spaces and let $f : X \to Y$. We say f is *uniformly continuous* if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$.

An *isometry* is a function $f : X \to Y$ such that $d(f(x), f(y)) = d(x, y)$ for each $x, y \in X$. Clearly an isometry is uniformly continuous.

Let X be a metric space with metric d. A sequence $(x_n)_{n\geq 1}$ in X is said to be a *Cauchy sequence* if for each $\epsilon > 0$ there exists N such that $n \geq N$ and $m \geq N$ implies $d(x_n, x_m) < \epsilon$. Obviously a convergent sequence is Cauchy.

Given any metric space X there is a complete metric space Y such that X is isometric to a dense subset of Y. One popular proof is to define Y in terms of equivalence classes of Cauchy sequences in X. See if you can work out an argument, or see [2], chapter 2, section 7, theorem 4. The space Y is essentially unique and is called the *completion* of X.

Exercise 40.2. *If the Cauchy sequence* $(x_n)_{n\geq 1}$ *has a convergent subsequence then it converges.*

Let X be a metric space. A subset $A \subset X$ is said to be *complete* if each Cauchy sequence in A is convergent to a point in A.

Exercise 40.3. *Let* X *be a metric space. Show each complete subset of* X *is closed. If* X *is complete show each closed subset is complete.*

Exercise 40.4. *We know* $(\mathbb{R}, d_1)$ *is complete. Show that* $(\mathbb{R}, d_2)$ *is not complete. (The metrics* d_1 *abd* d_2 *are defined above.)*

References

[1] Felix Hausdorff. *Grundzüge der Mengenlehre*. (Chelsea, New York, 1949), Leipzig (Revised 1944), 1914.

[2] A. N. Kolmogorov and S. V. Fomin. *Introductory Real Analysis*. Dover Publications, Inc., New York, 1975. Tranlated from Russian, edited and revised by Richard A. Silverman.

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Index