

60. Contraction Mappings

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The Contraction Mapping Principle was formulated by Banach in his 1920 thesis, published as [1] in 1922 (see page 160). The proof that Banach gave is the same as the one we give below, except that Banach works in a normed space and makes use of the linear structure.

There is a nice discussion of the Contraction Mapping Principle, and some applications, in Kolmogorov and Fomin, [2].

Given a function $f: X \rightarrow X$ a *fixed point* of f is a point $x \in X$ such that $f(x) = x$. A well-known topological theorem asserting the existence of at least one fixed point is the Brouwer theorem,

Theorem 60.1 (Brouwer). *If B is a closed ball in \mathbb{R}^n and $f: B \rightarrow B$ is a continuous map then f has a fixed point in B .*

Exercise 60.1. *Prove the Brouwer theorem in dimension 1, that is show if $f: [a, b] \rightarrow [a, b]$ is a continuous function then f has a fixed point. **Hint:** Consider the function g defined by $g(x) = x - f(x)$ and by looking at sign changes show that g must have a root in $[a, b]$.*

Since the Brouwer theorem deals with a finite dimensional situation it is not of much importance in functional analysis. However, the *Contraction Mapping Principle* which we study in the present section applies in many function spaces. In particular, as we will see, it implies the initial value problem for differential equations, under mild hypotheses, has a unique solution.

Let X be a metric space. A mapping $T: X \rightarrow X$ is called *contraction map* if there exists a constant c with $0 \leq c < 1$ such that

$$d(T(x), T(y)) \leq c d(x, y), \quad x, y \in X.$$

The constant c is called the *contractivity coefficient*.

Theorem 60.2 (Contraction Mapping Principle). Let X be a complete metric space and let $T : X \rightarrow X$ be a contraction mapping with contractivity coefficient c . Let $x_0 \in X$ and inductively define

$$x_{n+1} = T(x_n), \quad n \geq 0.$$

The T has a unique fixed point a , the sequence x_n converges to a and

$$d(a, x_n) \leq c^n d(a, x_0).$$

Proof. If $n \geq 1$ then $d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq c d(x_{n-1}, x_n)$ and so by induction $d(x_n, x_{n+1}) \leq c^n d(x_0, x_1)$. It follows if $0 \leq n < m$ then

$$d(x_n, x_m) \leq (c^n + \cdots + c^{m-1}) d(x_0, x_1) \leq \frac{c^n}{1-c} d(x_0, x_1)$$

and therefore (x_n) is a Cauchy sequence. Since X is complete this sequence converges to a point a . Now $T(a) = a$ by continuity of T .

If a is a fixed point of T , that is, $T(a) = a$ then $d(a, x_n) = d(T(a), T(x_{n-1})) \leq c d(a, x_{n-1})$. By induction we obtain $d(a, x_n) \leq c^n d(a, x_0)$.

For uniqueness note if a and b are fixed points then $d(a, b) = d(T(a), T(b)) \leq c d(a, b)$ which implies $d(a, b) = 0$ since $0 \leq c < 1$. \square

Note taking the limit as $m \rightarrow \infty$ in the inequality in the proof of the theorem we obtain

$$d(a, x_n) \leq \frac{c^n}{1-c} d(x_0, x_1)$$

which is sometimes more convenient than the inequality stated in the theorem.

The contraction mapping principle is useful, but the hypotheses are very strong and may be difficult to satisfy. If we examine the proof of the contraction mapping principle we are led to a simple result which may be more useful.

Theorem 60.3 (Picard iteration). Let X be a complete metric space and let $T : X \rightarrow X$ be a continuous map. Let $x_0 \in X$ and inductively define

$$x_{n+1} = T(x_n).$$

If

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$$

then the sequence $(x_n)_{n \geq 0}$ converges to a fixed point a of T . Moreover

$$d(a, x_n) \leq \sum_{k=n}^{\infty} d(x_k, x_{k+1}).$$

Proof. By the triangle inequality if $0 \leq n < m$ then

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}).$$

Thus $(x_n)_{n \geq 0}$ is a Cauchy sequence. If the limit is a then by continuity of T we have $a = T(a)$. Taking the limit as $m \rightarrow \infty$ in the inequality above we obtain the required estimate. \square

References

- [1] Stefan Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3:133–181, 1922.
- [2] A. N. Kolmogorov and S. V. Fomin. *Introductory Real Analysis*. Dover Publications, Inc., New York, 1975. Translated from Russian, edited and revised by Richard A. Silverman.

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Index

Brouwer fixed point theorem, 1

contraction map, 1

Contraction Mapping Principle, 2

contractivity coefficient, 1

fixed point, 1

Picard iteration theorem, 2

theorem

 Brouwer, 1

 Contraction Mapping Principle, 2

 Picard iteration, 2