100. Weyl Quantization

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In this note we describe informally pseudo-differential operators from the point of view of constructing an operational calculus for certain noncommuting operators, the operators corresponding to the position and momentum operators of quantum mechanics. All the arguments are formal yet we obtain a description of Weyl's quantization, [4], p. 274, in terms of pseudo-differential operator language, which shows it to be a symmetric compromise between natural choices.

Let $a, b \in \mathcal{O}_{\mathrm{M}}$. Since multiplication by a and b is continuous on the Schwartz space $\mathcal{S}$ we may define continuous linear operators

$$\begin{aligned} a(X)\colon \mathcal{S} \to \mathcal{S}, &\qquad a(X)u = au \\ b(-\mathrm{i}D)\colon \mathcal{S} \to \mathcal{S}, &\qquad (b(-\mathrm{i}D)u)\widehat{\ } = b\widehat{u}. \end{aligned}$$

With this notation X_k is multiplication by the coordinate function x_k and D_k is differentiation with respect to the coordinate function x_k and we have created a simple operational calculus for the operators X_k and for the operators D_k. If we try to extend our operational calculus to the full $2n$ operators we have a problem since they do not commute,

$$D_k X_k - X_k D_k = I.$$

How then do we define $p(X, \mathrm{i}D)$ for a suitable function p?

If we think of differentiating first, and then multiplying, we are led formally to the definition

$$\begin{aligned} p(\overset{2}{X}, -\mathrm{i}\overset{1}{D})u(x) &= (2\pi)^{-n} \int \mathrm{e}^{\mathrm{i}<\xi,x>} p(x,\xi)\widehat{u}(\xi)\ \mathrm{d}\xi \\ &= (2\pi)^{-n} \iint \mathrm{e}^{\mathrm{i}<\xi,x-y>} p(x,\xi)u(y)\ \mathrm{d}y\,\mathrm{d}\xi, \end{aligned}$$

where the overset numbers indicate the order. Alternately, if we multiply first and then differentiate we have

$$p(\overset{1}{X}, -\mathrm{i}\overset{2}{D})u(x) = (2\pi)^{-n} \iint \mathrm{e}^{\mathrm{i}<\xi,x-y>} p(y,\xi)u(y)\ \mathrm{d}y\,\mathrm{d}\xi,$$

The first operator order is useful for dealing with partial differential operators given in the the usual form

$$P = \sum_{\alpha} \mathrm{i}^{-|\alpha|} a_\alpha D^\alpha,$$

and generalizations, and the second operator order is useful for the transpose of such operators and generalizations. To deal with compositions we have to extend our operational calculus to functions of more variables and deal with more orders. It is natural to ask if there is a more symmetric approach.

In Hermann Weyl's book, [4], p. 275, he describes a rule for quantization of classical observables, functions p of x and ξ as above. The quantum observables are then suitable operators $p(X, -\mathrm{i}D)$. Weyl's approach is to use the formal Fourier inversion formula for $p(x, \xi)$ in terms of its Fourier transform $\widetilde{p}(\eta, y)$.

The Weyl quantization by means of the Fourier transform may may be used to construct an operational calculus for any $n-$tuple $A = (A_1, A_2, \cdots, A_n)$ operators on a given space. If we have a function f of n variables $x_1, \cdots, x_n$ then we have the formal Fourier transform and formal inverse transform

$$\begin{aligned} \widehat{f}(\xi) &= \int \mathrm{e}^{-\mathrm{i}<\xi,x>} f(x)\,\mathrm{d}x \\ f(x) &= (2\pi)^{-n} \int \mathrm{e}^{\mathrm{i}<\xi,x>} \widehat{f}(\xi)\,\mathrm{d}\xi. \end{aligned}$$

We define $f(A)$ by substituting A_k for x_k in the inverse transform,

$$f(A) = (2\pi)^{-n} \int \mathrm{e}^{\mathrm{i}<\xi,A>} \widehat{f}(\xi)\,\mathrm{d}\xi.$$

Here the operator $\mathrm{e}^{\mathrm{i}<\xi,A>}$ is defined in any one of the usual ways of defining the exponential of a suitable operator. See Robert F. V. Anderson, [1], for a treatment of this theory for operators on Banach space.

Let's return to the Weyl quantization. Here are the Fourier transform and inverse transform:

$$\begin{aligned} \widetilde{p}(\eta, y) &= \iint \mathrm{e}^{-\mathrm{i}<\eta,x>-\mathrm{i}<y,\xi>} p(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ p(x,\xi) &= (2\pi)^{-2n} \iint \mathrm{e}^{\mathrm{i}<\eta,x>+\mathrm{i}<y,\xi>} \widetilde{p}(\eta, y)\,\mathrm{d}\eta\,\mathrm{d}y. \end{aligned}$$

If we substitute X for x and $-\mathrm{i}D$ for ξ we obtain the Weyl operator W_p defined by

$$W_p = (2\pi)^{-2n} \iint \mathrm{e}^{T(\eta,y)} \widetilde{p}(\eta, y)\,\mathrm{d}\eta\,\mathrm{d}y,$$

where T is a family of first order differential operators given by

$$T(\eta, y) = \mathrm{i} < \eta, X > + < y, D > .$$

Since

$$[< y, D >, \mathrm{i} < \eta, X >] = \mathrm{i} < y, \eta >$$

we have to be careful when computing $\mathrm{e}^{T(\eta,y)}$. Let ϕ be the integrating factor for T, that is,

$$T(\eta, y)u = \mathrm{e}^{-\phi} < y, D > \left(\mathrm{e}^{\phi} u\right) .$$

This requires that

$$< y, D > \phi(x) = \mathrm{i} < \eta, x >$$

and a quick calculation yields

$$\phi(x) = \mathrm{i} \,|\, y \,|^{-2} < \eta, x >< y, x > - \frac{\mathrm{i}}{2} \,|\, y \,|^{-4} < y, x >^2 < \eta, y > .$$

From the definition of the integrating factor we have

$$\mathrm{e}^{T(\eta,y)} u = \mathrm{e}^{-\phi} \mathrm{e}^{<y,D>} \left(\mathrm{e}^{\phi} u\right) .$$

The exponential $\mathrm{e}^{<y,D>}$ is easily seen to be the translation operator τ_{-y}. Therefore

$$\left(\mathrm{e}^{T(\eta,y)} u\right)(x) = \mathrm{e}^{\phi(x+y)-\phi(x)} u(x + y).$$

Some simple algebra shows

$$\phi(x + y) - \phi(x) = \mathrm{i} < \eta, x + \frac{y}{2} >= \frac{\mathrm{i}}{2} < \eta, y > +\mathrm{i} < \eta, x >$$

and therefore

$$\boxed{\mathrm{e}^{\mathrm{i}<\eta,X>+<y,D>} = \mathrm{e}^{\frac{\mathrm{i}}{2}<\eta,y>} \mathrm{e}^{\mathrm{i}<\eta,X>} \mathrm{e}^{<y,D>}}$$

This formula is a very nice example of the breakdown of the group property of the exponential in the case of noncommuting variables. Such problems are usually studied with the help of the Campbell-Hausdorff formulae, see [2].

Let's return to calculating the Weyl operator. From our calculation of e^T above we have

$$\begin{aligned} W_p u(x) &= (2\pi)^{-2n} \iint \widetilde{p}(\eta, y) \mathrm{e}^{\mathrm{i}<\eta+\frac{y}{2}>} u(x + y)\ \mathrm{d}y\, \mathrm{d}\eta \\ &= (2\pi)^{-n} \iint \widehat{p}\left(x + \frac{y}{2}, y\right) u(x + y)\ \mathrm{d}y \end{aligned}$$

where we have inverted the Fourier transform relative to the first variables, so $\widehat{p}$ denotes the Fourier transform of p relative to the second set of variables.

Inserting the definition of the Fourier transform $\widehat{p}$ we have

$$\begin{aligned} W_p u(x) &= (2\pi)^{-n} \iint \mathrm{e}^{-\mathrm{i}<\xi,y>} p\left(x + \frac{y}{2}, \xi\right) u(x + y)\ \mathrm{d}y\, \mathrm{d}\xi \\ &= (2\pi)^{-n} \iint \mathrm{e}^{\mathrm{i}<\xi,x-y>} p\left(\frac{x + y}{2}, \xi\right) u(y)\ \mathrm{d}y\, \mathrm{d}\xi. \end{aligned}$$

This last expression shows how the Weyl operator is a nice symmetric compromise between multiplying first, and then differentiating, and differentiating first and then multiplying.

Exercise 1. *If* $p(x, \xi) = \mathrm{i} x_k \xi_k$ *then* $W_p u = \frac{1}{2} u + x_k D_k u = -\frac{1}{2} u + D_k(x_k u)$.

A good reference for the Weyl quantization from the pseudo-differential operators point of view is Hörmander, [3].

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References

[1] Robert F. V. Anderson. The Weyl functional calculus. *J. Functional Analysis*, 4:240–267, 1969.

[2] G. Hochschild. *The Structure of Lie Groups*. Holden-Day, San Francisco, 1965.

[3] Lars Hörmander. The Weyl calculus of pseudo-differential operators. *Comm. Pure Appl. Math.*, 32:359–443, 1979.

[4] Hermann Weyl. *The Theory of Groups and Quantum Mechanics (Gruppentheorie und Quantenmechnik)*. Dover, New York, 2 edition, 1950 (1928, 1930).