Can’t you just feel the moonshine?

Ken Ono (Emory University)
In my mind I'm going to Carolina. Can't you see the sunshine, can't you just feel the moonshine? Ain't it just like a friend of mine, to hit me from behind, and I'm goin' to Carolina in my mind.

— James Taylor —

AZ QUOTES
This talk

I. History of Moonshine: Tale of Two Research Programs

II. Distribution of Monstrous Moonshine (aka Witten’s Question)

III. New Moonshine
History of Finite Simple Groups

- (1832) Galois finds $A_n \ (n \geq 5)$ and $\text{PSL}_2(\mathbb{F}_p) \ (p \geq 5)$. 
(1832) Galois finds $A_n$ ($n \geq 5$) and $\text{PSL}_2(\mathbb{F}_p)$ ($p \geq 5$).

(1861-1873) Mathieu finds $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$ and $M_{24}$. 

(1832) Galois finds $A_n$ ($n \geq 5$) and $\text{PSL}_2(\mathbb{F}_p)$ ($p \geq 5$).

(1861-1873) Mathieu finds $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$ and $M_{24}$.

(1893) Cole classifies all simple groups with order $\leq 660$. 

I. History of Moonshine: Tale of Two Research Programs

Finite Groups

History of Finite Simple Groups

- (1832) Galois finds $A_n$ ($n \geq 5$) and $\text{PSL}_2(\mathbb{F}_p)$ ($p \geq 5$).
- (1861-1873) Mathieu finds $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$ and $M_{24}$.
- (1893) Cole classifies all simple groups with order $\leq 660$.
- (1890s-1972)
  Brauer, Burnside, Feit, Frobenius, Dickson, Hall, Thompson,.....
History of Finite Simple Groups

- (1832) Galois finds $A_n$ ($n \geq 5$) and $\text{PSL}_2(\mathbb{F}_p)$ ($p \geq 5$).
- (1861-1873) Mathieu finds $M_{11}, M_{12}, M_{22}, M_{23}$ and $M_{24}$.
- (1893) Cole classifies all simple groups with order $\leq 660$.
- (1890s-1972) Brauer, Burnside, Feit, Frobenius, Dickson, Hall, Thompson,.....
- (1972-1983: Gorenstein Program) Aschbacher, Fischer, Glauberman, Gorenstein, Greiss, Tits,.....
I. History of Moonshine: Tale of Two Research Programs

History of Finite Simple Groups

- (1832) Galois finds $A_n$ ($n \geq 5$) and $\text{PSL}_2(F_p)$ ($p \geq 5$).
- (1861-1873) Mathieu finds $M_{11}, M_{12}, M_{22}, M_{23}$ and $M_{24}$.
- (1893) Cole classifies all simple groups with order $\leq 660$.
- (1890s-1972) Brauer, Burnside, Feit, Frobenius, Dickson, Hall, Thompson,.....
- (1972-1983: Gorenstein Program) Aschbacher, Fischer, Glauberman, Gorenstein, Greiss, Tits,.....
- (1983) “Classification” announced with much fanfare.
The Monster

Conjecture (Fischer and Griess (1973))

There is a huge simple group $\mathbb{M}$ with order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$
Conjecture (Fischer and Griess (1973))

*There is a huge simple group* \( \mathbb{M} \) *with order*

\[
2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.
\]

Theorem (Griess (1982))

*The Monster group* \( \mathbb{M} \) *exists.*
I. History of Moonshine: Tale of Two Research Programs

Finite Groups

The Monster

Conjecture (Fischer and Griess (1973))

There is a huge simple group \( \mathbb{M} \) with order

\[
2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.
\]

Theorem (Griess (1982))

The Monster group \( \mathbb{M} \) exists.

In particular, it is the automorphism gp of an explicit commutative non-associative algebra on a 196884 dim \( \mathbb{R} \)-vector space.
Finite simple groups live in natural infinite families, apart from 26 sporadic groups.
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Finite Groups

The Monster’s Happy Family

The subgroups and sub-quotients of $M$ are a Happy Family.
I. History of Moonshine: Tale of Two Research Programs

Modular curves

Can't you just feel the moonshine?
I. History of Moonshine: Tale of Two Research Programs

Modular curves

Facts

1. \( \text{SL}_2(\mathbb{Z}) \) acts on the upper-half complex plane \( \mathbb{H} \) by

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \quad \leftrightarrow \quad \gamma \tau \mapsto \frac{a \tau + b}{c \tau + d}
\]
I. History of Moonshine: Tale of Two Research Programs

Modular curves

Facts

1. \( \text{SL}_2(\mathbb{Z}) \) acts on the upper-half complex plane \( \mathbb{H} \) by

\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \quad \iff \quad \gamma \tau \mapsto \frac{a \tau + b}{c \tau + d} \]

2. For congruence subgroups \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \), number theorists are interested in the quotients

\[ Y(\Gamma) := \Gamma \backslash \mathbb{H}. \]
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Modular curves

Facts

1. $\text{SL}_2(\mathbb{Z})$ acts on the upper-half complex plane $\mathbb{H}$ by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \quad \leftrightarrow \quad \gamma \tau \mapsto \frac{a\tau + b}{c\tau + d}$$

2. For congruence subgroups $\Gamma \subset \text{SL}_2(\mathbb{Z})$, number theorists are interested in the quotients

$$Y(\Gamma) \ := \ \Gamma \backslash \mathbb{H}.$$ 

3. These may be compactified by “adding cusps” to obtain compact Riemann surfaces, the modular curves $X(\Gamma)$. 

Modular curves
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Modular curves

\[ \Gamma = \text{SL}_2(\mathbb{Z}) \]
I. History of Moonshine: Tale of Two Research Programs

Modular curves

\[ \Gamma = \text{SL}_2(\mathbb{Z}) \]

Remark

\( X_0(1) \) has genus 0, which implies that its field of modular functions is \( \mathbb{C}(j(\tau)) \) with a Hauptmodul \( j(\tau) \).
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Modular curves

Modular functions

**Definition**

A meromorphic function \( f : \mathbb{H} \rightarrow \mathbb{C} \) is a \( \Gamma \)-modular function if for every \( \gamma \in \Gamma \) we have

\[
f(\gamma \tau) = f(\tau).
\]
Modular functions

**Definition**

A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a $\Gamma$-modular function if for every $\gamma \in \Gamma$ we have

$$f(\gamma \tau) = f(\tau).$$

**Example ($\Gamma = \text{SL}_2(\mathbb{Z})$)**

The **Hauptmodul** is Klein’s $j$-function ($q := e^{2\pi i \tau}$)

$$j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n) q^n$$

$$= q^{-1} + 196884 q + 21493760 q^2 + 864299970 q^3 + \ldots$$
Elliptic functions and curves

Fundamental Facts (Classical)

1. If $\Lambda$ is a rank 2 lattice in $\mathbb{C}$, then $\mathbb{C}/\Lambda$ is an elliptic curve.
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Modular curves

Elliptic functions and curves

Fundamental Facts (Classical)

1. If $\Lambda$ is a rank 2 lattice in $\mathbb{C}$, then $\mathbb{C}/\Lambda$ is an elliptic curve.

2. If $\Lambda_{\tau} := \mathbb{Z} \oplus \mathbb{Z} \tau$, then $j(\tau)$ is an invariant of the elliptic curve.
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Modular curves

Elliptic functions and curves

Fundamental Facts (Classical)

1. If $\Lambda$ is a rank 2 lattice in $\mathbb{C}$, then $\mathbb{C}/\Lambda$ is an elliptic curve.

2. If $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$, then $j(\tau)$ is an invariant of the elliptic curve.

Remarks

1. i.e. $X_0(1)$ encode isomorphism classes of elliptic curves.
Elliptic functions and curves

Fundamental Facts (Classical)

1. If $\Lambda$ is a rank 2 lattice in $\mathbb{C}$, then $\mathbb{C}/\Lambda$ is an **elliptic curve**.
2. If $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$, then $j(\tau)$ is an invariant of the elliptic curve.

Remarks

1. *i.e.* $X_0(1)$ encode isomorphism classes of elliptic curves.
2. *Congruence modular curves* $X(\Gamma)$ encodes isomorphism classes of elliptic curves with **extra structure** (*cf.* Mazur’s Thm).
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Modular curves

Modularity of Elliptic curves

Theorem (Taylor-Wiles, et. al. (1990s))

*Every elliptic curve over \( \mathbb{Q} \) has an \( X_0(N) \) parametrization.*
I. History of Moonshine: Tale of Two Research Programs

Modular curves

Hyperelliptic Modular Curves
Hyperelliptic Modular Curves

Theorem (Ogg (1974))

$X_0(N)$ est hyperelliptique pour exactement dix-neuf valuers de $N$. 
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Modular curves

Hyperelliptic Modular Curves

Theorem (Ogg (1974))

\[ X_0(N) \text{ est hyperelliptique pour exactement dix-neuf valuers de } N. \]

Problem

An elliptic curve \( E \) over an algebraic closure \( \overline{\mathbb{F}}_p \) is supersingular if it has no \( p \)-torsion.
I. History of Moonshine: Tale of Two Research Programs

Modular curves

Hyperelliptic Modular Curves

Theorem (Ogg (1974))

\(X_0(N)\) est hyperelliptique pour exactement dix-neuf valuers de \(N\).

Problem

An elliptic curve \(E\) over an algebraic closure \(\overline{\mathbb{F}}_p\) is supersingular if it has no \(p\)-torsion.

For which \(p\) is it true that every supersingular \(E\) lives over \(\mathbb{F}_p\)?
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs
Modular curves

First Hint of Moonshine

Corollary (Ogg (1974))

\[ Toutes \ les \ valuers \ supersingulières \ de \ j \ sont \ \mathbb{F}_p \ si, \ et \ seulement \ si \]
\[ p \in Ogg_{ss} := \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}. \]
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Modular curves

First Hint of Moonshine

Corollary (Ogg (1974))

\[ \text{Toutes les valuers supersingulières de } j \text{ sont } \mathbb{F}_p \text{ si, et seulement si} \]

\[ p \in Ogg_{ss} := \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}. \]

Remarque 1. — Dans sa leçon d’ouverture au Collège de France, le 14 janvier 1975, J. Tits mentionna le groupe de Fischer, "le monstre", qui, s’il existe, est un groupe simple "sporadique" d’ordre

\[ 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71, \]

i.e. divisible exactement par les quinze nombres premiers de la liste du corollaire. Une bouteille de Jack Daniels est offerte à celui qui expliquera cette coïncidence.
John McKay observed that

\[ 196884 = 1 + 196883 \]
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Monstrous Moonshine

John Thompson’s generalizations

Thompson further observed:

\[
\begin{align*}
196884 &= 1 + 196883 \\
21493760 &= 1 + 196883 + 21296876 \\
864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326
\end{align*}
\]

Coefficients of \( j(\tau) \) — Dimensions of irreducible representations of the Monster \( \mathbb{M} \)
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Monstrous Moonshine

John Thompson’s generalizations

Thompson further observed:

\[ 196884 = 1 + 196883 \]
\[ 21493760 = 1 + 196883 + 21296876 \]
\[ 864299970 = 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \]

Coefﬁcients of \( j(\tau) \)

Dimensions of irreducible representations of the Monster \( \mathbb{M} \)

Remark

Klein’s \( j \)-function

\[ j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n)q^n \]
\[ = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \ldots. \]
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Monstrous Moonshine

The Monster characters

The character table for $\mathbb{M}$ (ordered by size) gives dimensions:

\[\chi_1(e) = 1\]
\[\chi_2(e) = 196883\]
\[\chi_3(e) = 21296876\]
\[\chi_4(e) = 842609326\]
\[
\vdots
\]
\[\chi_{194}(e) = 258823477531055064045234375.\]
McKay and Thompson

Conjecture (Thompson)

*There is an infinite-dimensional graded module* \( V^\text{\#} = \bigoplus_{n=-1}^{\infty} V_n^\text{\#} \)
*for which* \( \dim(V_n^\text{\#}) = c(n) \).
I. History of Moonshine: Tale of Two Research Programs

McKay and Thompson

**Conjecture (Thompson)**

*There is an infinite-dimensional graded module* \( V^q = \bigoplus_{n=-1}^{\infty} V_n^q \)
*for which* \( \dim(V_n^q) = c(n) \).

**Definition (Thompson)**

Assuming the conjecture, if \( g \in \mathbb{M} \), then define the McKay–Thompson series

\[
T_g(\tau) := \sum_{n=-1}^{\infty} \text{Tr}(g|V_n^q)q^n.
\]
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs
Monstrous Moonshine

Conway and Norton

Conjecture (Monstrous Moonshine, 1979)

For each $g \in \mathbb{M}$ there is an explicit genus 0 congruence subgroup $\Gamma_g \subset \text{SL}_2(\mathbb{R})$ for which $T_g(\tau)$ is the **Hauptmodul**.
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

Monstrous Moonshine

Borcherds’ work

Theorem (Frenkel–Lepowsky–Meurman (1980s))

If it exists, then the moonshine module $V_{\mathbb{H}} = \bigoplus_{n=-1}^{\infty} V_n^{\mathbb{H}}$ is a specific vertex operator algebra whose automorphism group is $\mathbb{M}$. 
I. History of Moonshine: Tale of Two Research Programs

Monstrous Moonshine

Borcherds’ work

Theorem (Frenkel–Lepowsky–Meurman (1980s))

If it exists, then the moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ is a specific vertex operator algebra whose automorphism group is $\mathbb{M}$.

Theorem (Borcherds (1998 Fields Medal))

The Monstrous Moonshine Conjecture is true.
Question A

Do order \( p \) elements in \( \mathbb{M} \) know the \( \overline{F}_p \) supersingular \( j \)-invariants?
Can't you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

The Jack Daniels Problem

Question A

*Do order p elements in $\mathbb{M}$ know the $\overline{F}_p$ supersingular $j$-invariants?*

Question B

*If $p \not\in \text{Ogg}_{ss}$, then why is it true that $p \nmid \#\mathbb{M}$?*
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

The Jack Daniels Problem

Question A

Do order $p$ elements in $\mathbb{M}$ know the $\overline{F}_p$ supersingular $j$-invariants?

Question B

If $p \not\in Ogg_{ss}$, then why is it true that $p \nmid \# \mathbb{M}$?

Question C

If $p \in Ogg_{ss}$, then why do we know (a priori) that $p \mid \# \mathbb{M}$?
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs
The Jack Daniels Problem

Questions A and B

- Question A is answered by combining the “group law interpreted in moonshine” with work of Dwork.
Can’t you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs
   The Jack Daniels Problem

Questions A and B

- Question A is answered by combining the “group law interpreted in moonshine” with work of Dwork.
- Question B is answered by the proof of Monstrous Moonshine.
Ogg’s Problem

**Question C**

*If* $p \in \text{Ogg}_{ss}$, *then* why do we know (a priori) that $p \mid \#M$?"
Question C

*If* $p \in Ogg_{ss}$, *then why do we know (a priori) that* $p | \#M$?

Theorem 1 (Duncan-O (2016))

*The following are true:*
Ogg’s Problem

Question C

If \( p \in Ogg_{ss} \), then why do we know (a priori) that \( p \mid \#M \)?

Theorem 1 (Duncan-O (2016))

The following are true:

1. If \( p \in Ogg_{ss} \), then the Hauptmodul \( h_p(\tau) \) is spanned \( p \)-adically by elementary theta functions.
Can't you just feel the moonshine?

I. History of Moonshine: Tale of Two Research Programs

The Jack Daniels Problem

Ogg’s Problem

Question C

If \( p \in Ogg_{ss} \), then why do we know (a priori) that \( p \mid \#M \)?

Theorem 1 (Duncan-O (2016))

The following are true:

1. If \( p \in Ogg_{ss} \), then the Hauptmodul \( h_p(\tau) \) is spanned \( p \)-adically by elementary theta functions.

2. These spaces mod \( p \) are spanned by elementary theta functions iff \( p \mid \#M \).
Distribution of Monstrous Moonshine
Can’t you just feel the moonshine?
II. Distribution of Monstrous Moonshine

Witten’s Conjecture (2007)

Conjecture (Witten, Li-Song-Strominger)

The vertex operator algebra $V^\frac{1}{2}$ is dual to a 3d quantum gravity theory. Thus, there are 194 “black hole states".
Conjecture (Witten, Li-Song-Strominger)

The vertex operator algebra $V^\mathfrak{h}$ is dual to a 3d quantum gravity theory. Thus, there are 194 “black hole states”.

Question (Witten)

How are these different kinds of black hole states distributed?
Can’t you just feel the moonshine?

II. Distribution of Monstrous Moonshine

Open Problem

Question

Consider the moonshine expressions

\[ 196884 = 1 + 196883 \]
\[ 21493760 = 1 + 196883 + 21296876 \]
\[ 864299970 = 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \]
\[ \vdots \]

\[ c(n) = \sum_{i=1}^{194} m_i(n) \chi_i(e) \]

How many ‘1’s, ‘196883’s, etc. show up in these equations?
II. Distribution of Monstrous Moonshine

Some Proportions

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta(m_1(n))$</th>
<th>$\delta(m_2(n))$</th>
<th>$\ldots$</th>
<th>$\delta(m_{194}(n))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$\ldots$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
### Some Proportions

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta (m_1(n))$</th>
<th>$\delta (m_2(n))$</th>
<th>$\cdots$</th>
<th>$\delta (m_{194}(n))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td>$\cdots$</td>
<td>0</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>40</td>
<td>$4.011 \ldots \times 10^{-4}$</td>
<td>$2.514 \ldots \times 10^{-3}$</td>
<td>$\cdots$</td>
<td>0.00891\ldots</td>
</tr>
</tbody>
</table>
Can’t you just feel the moonshine?

II. Distribution of Monstrous Moonshine

Some Proportions

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta(m_1(n))$</th>
<th>$\delta(m_2(n))$</th>
<th>$\delta(m_3(n))$</th>
<th>$\delta(m_{194}(n))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>$4.011 \ldots \times 10^{-4}$</td>
<td>$2.514 \ldots \times 10^{-3}$</td>
<td></td>
<td>$0.00891 \ldots$</td>
</tr>
<tr>
<td>60</td>
<td>$2.699 \ldots \times 10^{-9}$</td>
<td>$2.732 \ldots \times 10^{-8}$</td>
<td></td>
<td>$0.04419 \ldots$</td>
</tr>
</tbody>
</table>
Can’t you just feel the moonshine?

II. Distribution of Monstrous Moonshine

Some Proportions

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta(m_1(n))$</th>
<th>$\delta(m_2(n))$</th>
<th>...</th>
<th>$\delta(m_{194}(n))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td></td>
<td>$0$</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td></td>
<td>:</td>
</tr>
<tr>
<td>40</td>
<td>$4.011 \ldots \times 10^{-4}$</td>
<td>$2.514 \ldots \times 10^{-3}$</td>
<td>...</td>
<td>$0.00891 \ldots$</td>
</tr>
<tr>
<td>60</td>
<td>$2.699 \ldots \times 10^{-9}$</td>
<td>$2.732 \ldots \times 10^{-8}$</td>
<td>...</td>
<td>$0.04419 \ldots$</td>
</tr>
<tr>
<td>80</td>
<td>$4.809 \ldots \times 10^{-14}$</td>
<td>$7.537 \ldots \times 10^{-13}$</td>
<td>...</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>100</td>
<td>$4.427 \ldots \times 10^{-18}$</td>
<td>$1.077 \ldots \times 10^{-16}$</td>
<td>...</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>120</td>
<td>$1.377 \ldots \times 10^{-21}$</td>
<td>$5.501 \ldots \times 10^{-20}$</td>
<td>...</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>140</td>
<td>$1.156 \ldots \times 10^{-24}$</td>
<td>$1.260 \ldots \times 10^{-22}$</td>
<td>...</td>
<td>$0.04428 \ldots$</td>
</tr>
</tbody>
</table>
II. Distribution of Monstrous Moonshine

Some Proportions

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta(m_1(n))$</th>
<th>$\delta(m_2(n))$</th>
<th>...</th>
<th>$\delta(m_{194}(n))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>40</td>
<td>$4.011\ldots \times 10^{-4}$</td>
<td>$2.514\ldots \times 10^{-3}$</td>
<td>...</td>
<td>$0.00891\ldots$</td>
</tr>
<tr>
<td>60</td>
<td>$2.699\ldots \times 10^{-9}$</td>
<td>$2.732\ldots \times 10^{-8}$</td>
<td>...</td>
<td>$0.04419\ldots$</td>
</tr>
<tr>
<td>80</td>
<td>$4.809\ldots \times 10^{-14}$</td>
<td>$7.537\ldots \times 10^{-13}$</td>
<td>...</td>
<td>$0.04428\ldots$</td>
</tr>
<tr>
<td>100</td>
<td>$4.427\ldots \times 10^{-18}$</td>
<td>$1.077\ldots \times 10^{-16}$</td>
<td>...</td>
<td>$0.04428\ldots$</td>
</tr>
<tr>
<td>120</td>
<td>$1.377\ldots \times 10^{-21}$</td>
<td>$5.501\ldots \times 10^{-20}$</td>
<td>...</td>
<td>$0.04428\ldots$</td>
</tr>
<tr>
<td>140</td>
<td>$1.156\ldots \times 10^{-24}$</td>
<td>$1.260\ldots \times 10^{-22}$</td>
<td>...</td>
<td>$0.04428\ldots$</td>
</tr>
<tr>
<td>160</td>
<td>$2.621\ldots \times 10^{-27}$</td>
<td>$3.443\ldots \times 10^{-23}$</td>
<td>...</td>
<td>$0.04428\ldots$</td>
</tr>
<tr>
<td>180</td>
<td>$1.877\ldots \times 10^{-28}$</td>
<td>$3.371\ldots \times 10^{-23}$</td>
<td>...</td>
<td>$0.04428\ldots$</td>
</tr>
<tr>
<td>200</td>
<td>$1.715\ldots \times 10^{-28}$</td>
<td>$3.369\ldots \times 10^{-23}$</td>
<td>...</td>
<td>$0.04428\ldots$</td>
</tr>
</tbody>
</table>
Can't you just feel the moonshine?

II. Distribution of Monstrous Moonshine

Some Proportions

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta(m_1(n))$</th>
<th>$\delta(m_2(n))$</th>
<th>$\ldots$</th>
<th>$\delta(m_{194}(n))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$\ldots$</td>
<td>$0$</td>
</tr>
<tr>
<td>...</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>40</td>
<td>$4.011 \ldots \times 10^{-4}$</td>
<td>$2.514 \ldots \times 10^{-3}$</td>
<td>$\ldots$</td>
<td>$0.00891 \ldots$</td>
</tr>
<tr>
<td>60</td>
<td>$2.699 \ldots \times 10^{-9}$</td>
<td>$2.732 \ldots \times 10^{-8}$</td>
<td>$\ldots$</td>
<td>$0.04419 \ldots$</td>
</tr>
<tr>
<td>80</td>
<td>$4.809 \ldots \times 10^{-14}$</td>
<td>$7.537 \ldots \times 10^{-13}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>100</td>
<td>$4.427 \ldots \times 10^{-18}$</td>
<td>$1.077 \ldots \times 10^{-16}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>120</td>
<td>$1.377 \ldots \times 10^{-21}$</td>
<td>$5.501 \ldots \times 10^{-20}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>140</td>
<td>$1.156 \ldots \times 10^{-24}$</td>
<td>$1.260 \ldots \times 10^{-22}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>160</td>
<td>$2.621 \ldots \times 10^{-27}$</td>
<td>$3.443 \ldots \times 10^{-23}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>180</td>
<td>$1.877 \ldots \times 10^{-28}$</td>
<td>$3.371 \ldots \times 10^{-23}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>200</td>
<td>$1.715 \ldots \times 10^{-28}$</td>
<td>$3.369 \ldots \times 10^{-23}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>220</td>
<td>$1.711 \ldots \times 10^{-28}$</td>
<td>$3.368 \ldots \times 10^{-23}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
<tr>
<td>240</td>
<td>$1.711 \ldots \times 10^{-28}$</td>
<td>$3.368 \ldots \times 10^{-23}$</td>
<td>$\ldots$</td>
<td>$0.04428 \ldots$</td>
</tr>
</tbody>
</table>
Can’t you just feel the moonshine?

II. Distribution of Monstrous Moonshine

Distribution of Moonshine

Theorem 2 (Duncan, Griffin, O (2015))

If \(1 \leq i \leq 194\), then as \(n \to +\infty\) we have

\[
\text{m}_i(n) \sim \frac{\text{dim}(\chi_i)}{\sqrt{2} |n|^{3/4} |\mathbb{M}|} \cdot e^{4\pi \sqrt{|n|}}
\]

Corollary (Duncan, Griffin, O)

The Moonshine module is asymptotically regular. In other words, we have

\[
\delta(m_i) := \lim_{n \to +\infty} \frac{m_i(n)}{\sum_{1 \leq i \leq 194} m_i(n)} = \text{dim}(\chi_i)
\]

5844076785304502808013602136
Can’t you just feel the moonshine?

II. Distribution of Monstrous Moonshine

Distribution of Moonshine

Theorem 2 (Duncan, Griffin, O (2015))

If $1 \leq i \leq 194$, then as $n \to +\infty$ we have

$$m_i(n) \sim \frac{\dim(\chi_i)}{\sqrt{2|n|^{3/4}|\mathcal{M}|}} \cdot e^{4\pi\sqrt{|n|}}.$$

Corollary (Duncan, Griffin, O)

The Moonshine module is asymptotically regular.

In other words, we have

$$\delta (m_i) := \lim_{n \to +\infty} \frac{m_i(n)}{\sum_{i=1}^{194} m_i(n)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.$$
II. Distribution of Monstrous Moonshine

Proof

Orthogonality

Fact

If $G$ is a group and $g, h \in G$, then

$$
\sum_{\chi_i} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} 
|C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate} \\
0 & \text{otherwise},
\end{cases}
$$

where $C_G(g)$ is the centralizer of $g$ in $G$. 

Can’t you just feel the moonshine?
II. Distribution of Monstrous Moonshine

Proof

Orthogonality

Fact

If $G$ is a group and $g, h \in G$, then

$$\sum_{\chi_i} \chi_i(g) \overline{\chi_i(h)} = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise,} \end{cases}$$

where $C_G(g)$ is the centralizer of $g$ in $G$.

From this we can work out that

$$T_{\chi_i}(\tau) = \sum_{n=-1}^{\infty} m_i(n) q^n.$$
II. Distribution of Monstrous Moonshine

Proof

Exact Formulas Imply Distributions

Theorem 3 (Duncan, Griffin,O (2015))

We have (ugly) exact formulas for the coefficients of the $T_{\chi_i}(\tau)$. 
II. Distribution of Monstrous Moonshine

Proof

Exact Formulas Imply Distributions

**Theorem 3 (Duncan, Griffin, O (2015))**

*We have (ugly) exact formulas for the coefficients of the $T_{\chi_i}(\tau)$.*

**Sketch of Proof**

*Reduces to a “Kloostermania” problem in the spectral theory of automorphic forms.*
Can’t you just feel the moonshine?

III. New Moonshines

Umbral Moonshine

Umbral (shadow) Moonshine
Can’t you just feel the moonshine?

III. New Moonshines

Umbral Moonshine

An unexpected observation

Observation (Eguchi, Ooguri, Tachikawa (2010))

*Using the K3 surface elliptic genus, there is a mock modular form*

\[ H(\tau) = q^{-\frac{1}{8}} \left( -2 + 45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \ldots \right). \]
Observation (Eguchi, Ooguri, Tachikawa (2010))

Using the K3 surface elliptic genus, there is a mock modular form

\[ H(\tau) = q^{-\frac{1}{8}} \left( -2 + 45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \ldots \right). \]

The degrees of the irreducible repn’s of the Mathieu group \( M_{24} \) are:

1, 23, 45, 231, 252, 253, 483, 770, 990, 1035,
1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395.
Mathieu Moonshine

Theorem (Gannon (2013))

There is an infinite dimensional graded $M_{24}$-module whose McKay-Thompson series are specific mock modular forms.
What are mock modular forms?

**Notation.** Throughout, let

\[ \tau = x + iy \in \mathbb{H} \text{ with } x, y \in \mathbb{R}. \]
What are mock modular forms?

**Notation.** Throughout, let

\[ \tau = x + iy \in \mathbb{H} \text{ with } x, y \in \mathbb{R}. \]

**Hyperbolic Laplacian.**

\[ \Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \]
Harmonic Maass forms

**Definition**

A **harmonic Maass form of weight** $k$ **on a subgroup**

$\Gamma \subset SL_2(\mathbb{Z})$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying:

1. For all $A = \left( \begin{array}{ll} a & b \\ c & d \end{array} \right) \in \Gamma$ and $\tau \in \mathbb{H}$, we have
   
   $$M \left( a\tau + b \right) c \tau + d \right) = \left( c \tau + d \right)^k M(\tau).$$

2. We have that $\Delta_k M = 0$.

**Remark**

Modular forms are a density 0 subset of harmonic Maass forms.
Harmonic Maass forms

**Definition**

A harmonic Maass form of weight $k$ on a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying:

1. For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}$, we have

   $$M \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k \ M(\tau).$$
Harmonic Maass forms

**Definition**

A harmonic Maass form of weight $k$ on a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying:

1. For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}$, we have
   
   $$M \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k \ M(\tau).$$

2. We have that $\Delta_k M = 0$. 

Remark

Modular forms are a density 0 subset of harmonic Maass forms.
Harmonic Maass forms

Definition

A harmonic Maass form of weight $k$ on a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying:

1. For all $A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$ and $\tau \in \mathbb{H}$, we have
   
   $$M \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k \ M(\tau).$$

2. We have that $\Delta_k M = 0$.

Remark

*Modular forms are a density 0 subset of harmonic Maass forms.*
Can’t you just feel the moonshine?

III. New Moonshines
Umbral Moonshine

Coming in 2017...

Harmonic Maass Forms and Mock Modular Forms: Theory and Applications

Kathrin Bringmann
Amanda Folsom
Ken Ono
Larry Rolen
III. New Moonshines
Umbral Moonshine

Fourier expansions ($q := e^{2\pi i \tau}$)

**Fundamental Lemma**

If $M \in H_{2-k}$ and $\Gamma(a, x)$ is the incomplete $\Gamma$-function, then

$$M(\tau) = \sum_{n \gg -\infty} c^+(n) q^n + \sum_{n < 0} c^-(n) \Gamma(k - 1, 4\pi |n| y) q^n.$$  

$$\updownarrow$$  

Holomorphic part $M^+$  
Nonholomorphic part $M^-$

Remark: We call $M^+$ a mock modular form if $M^- \neq 0$.  

If $\xi^2 - k := 2iy^2 - k$, then the shadow of $M$ is $\xi^2 - k(M - \tilde{M})$.  

Fourier expansions \((q := e^{2\pi i \tau})\)

**Fundamental Lemma**

If \(M \in H_{2-k}\) and \(\Gamma(a, x)\) is the incomplete \(\Gamma\)-function, then

\[
M(\tau) = \sum_{n \gg -\infty} c^+(n)q^n + \sum_{n < 0} c^-(n)\Gamma(k - 1, 4\pi|n|y)q^n.
\]

\[
\uparrow \quad \uparrow
\]

Holomorphic part \(M^+\) \quad Nonholomorphic part \(M^-\)

**Remark**

- *We call* \(M^+\) *a mock modular form if* \(M^- \neq 0\).*
Can’t you just feel the moonshine?

III. New Moonshines

Umbral Moonshine

Fourier expansions \( (q := e^{2\pi i \tau}) \)

**Fundamental Lemma**

*If \( M \in H_{2-k} \) and \( \Gamma(a, x) \) is the incomplete \( \Gamma \)-function, then*

\[
M(\tau) = \sum_{n \gg -\infty} c^+(n) q^n + \sum_{n < 0} c^-(n) \Gamma(k - 1, 4\pi |n|y) q^n.
\]

\[\uparrow\]

Holomorphic part \( M^+ \)

\[\uparrow\]

Nonholomorphic part \( M^- \)

**Remark**

- *We call \( M^+ \) a mock modular form if \( M^- \neq 0 \).*

- *If \( \xi_{2-k} := 2iy^{2-k} \frac{\partial}{\partial \tau} \), then the shadow of \( M \) is \( \xi_{2-k}(M^-) \).*
Can’t you just feel the moonshine?

III. New Moonshines
Umbral Moonshine

Ramanujan’s Deathbed Letter

Dear Hardy, January 12, 1920

I am extremely sorry for not writing you a single letter up to now...I discovered very interesting functions recently which I call “Mock” θ-functions...they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples....

Yours truly,

S. Ramanujan
Larger Framework of Moonshine?

Remark

There are well known connections with even unimodular positive definite rank 24 lattices:

\[ M \leftrightarrow \text{"Leech lattice"} \]

\[ M_{24} \leftrightarrow A_1^{24} \text{ lattice.} \]
Umbral Moonshine Conjecture

Conjecture (Cheng, Duncan, Harvey (2013))

Let $L^X$ be an even unimodular positive-def rank 24 lattice, and let:

- $X$ be its root system and $W^X$ its Weyl group.
Umbral Moonshine Conjecture

Conjecture (Cheng, Duncan, Harvey (2013))

Let $L^X$ be an even unimodular positive-def rank 24 lattice, and let :
- $X$ be its root system and $W^X$ its Weyl group.
- The umbral group $G^X := \text{Aut}(L^X)/W^X$. 
Umbral Moonshine Conjecture

Conjecture (Cheng, Duncan, Harvey (2013))

Let $L^X$ be an even unimodular positive-def rank 24 lattice, and let:

- $X$ be its root system and $W^X$ its Weyl group.
- The umbral group $G^X := \text{Aut}(L^X)/W^X$.

Then there is moonshine for $G^X$. 
Umbral Moonshine Conjecture

Conjecture (Cheng, Duncan, Harvey (2013))

Let $L^X$ be an even unimodular positive-def rank 24 lattice, and let:

- $X$ be its root system and $W^X$ its Weyl group.
- The umbral group $G^X := \text{Aut}(L^X)/W^X$.

Then there is moonshine for $G^X$.

Remark

Apart from the Leech case, the McKay-Thompson series are weight 1/2 mock modular forms with theta function shadows.
Can’t you just feel the moonshine?

III. New Moonshines
Umbral Moonshine

Our results.

Theorem 4 (Duncan, Griffin, O (2015))

*The Umbral Moonshine Conjecture is true.*
Our results.

Theorem 4 (Duncan, Griffin, O (2015))

The Umbral Moonshine Conjecture is true.

Example

For \( M_{12} \) the MT series include Ramanujan’s deathbed functions:

\[
\begin{align*}
  f(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 + q)^2(1 + q^2)^2 \cdots (1 + q^n)^2}, \\
  \phi(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 + q^2)(1 + q^4) \cdots (1 + q^{2n})}, \\
  \chi(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 - q + q^2)(1 - q^2 + q^4) \cdots (1 - q^n + q^{2n})}
\end{align*}
\]
Can’t you just feel the moonshine?

III. New Moonshines

Pariah Moonshine

Can’t we feel the Moonshine?
Can’t we feel the Moonshine?

Question

Moonshine is now “largely” understood in the following settings:

- The Monster $\mathbb{M}$ and its Happy Family.
Can’t we feel the Moonshine?

III. New Moonshines

Pariah Moonshine

Can’t you just feel the moonshine?

Question

Moonshine is now “largely” understood in the following settings:

- *The Monster* $\mathbb{M}$ and its Happy Family.
- Umbral Groups (automorphisms of nice rank 24 lattices).

Question (Conway-Norton, 1979)

“We ask whether the sporadic simple groups that may not be involved with $\mathbb{M}$ have moonshine properties.”
Can’t you just feel the moonshine?

III. New Moonshines
Pariah Moonshine

Can’t we feel the Moonshine?

Question

Moonshine is now “largely” understood in the following settings:

- The Monster $\mathbb{M}$ and its Happy Family.
- Umbral Groups (automorphisms of nice rank 24 lattices).

Is there more out there?
Can’t you just feel the moonshine?

III. New Moonshines

Pariah Moonshine

Can’t we feel the Moonshine?

Question

Moonshine is now “largely” understood in the following settings:

- The Monster $\mathbb{M}$ and its Happy Family.
- Umbral Groups (automorphisms of nice rank 24 lattices).

Is there more out there?

Question (Conway-Norton, 1979)

“We ask whether the sporadic simple groups that may not be involved with $\mathbb{M}$ have moonshine properties.”
Can’t you just feel the moonshine?

III. New Moonshines
Pariah Moonshine

Pariah Groups

Theorem (O’Nan (1976))
There is a sporadic finite group $ON$ with order

$$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$$
outside the Happy Family.
Can't you just feel the moonshine?

III. New Moonshines

Pariah Moonshine

Pariah Groups

Theorem (O’Nan (1976))

There is a sporadic finite group \( ON \) with order

\[
2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31 \text{ outside the Happy Family.}
\]
Can’t you just feel the moonshine?

III. New Moonshines
Pariah Moonshine

O’Nan Moonshine

Theorem 5 (Duncan, Mertens, O (2017))

There is an infinite dimensional graded ON moonshine module.
O’Nan Moonshine

Theorem 5 (Duncan, Mertens, O (2017))

There is an infinite dimensional graded ON moonshine module. Its MT series are explicit weight 3/2 mock modular forms.
O’Nan Moonshine

Theorem 5 (Duncan, Mertens, O (2017))

There is an infinite dimensional graded ON moonshine module. Its MT series are explicit weight 3/2 mock modular forms.

Remarks (Graded Dimensions)

1. If we let $W := \bigoplus_n W_n$, then

$$\dim W_n = \text{“traces of CM disc } - n \text{ values of } J_2\text{”}.$$
O’Nan Moonshine

Theorem 5 (Duncan, Mertens, O (2017))

There is an infinite dimensional graded ON moonshine module. Its MT series are explicit weight 3/2 mock modular forms.

Remarks (Graded Dimensions)

1. If we let $W := \oplus_n W_n$, then

\[
\dim W_n = \text{“traces of CM disc } - n \text{ values of } J_2\text{”}.
\]

2. We have

\[
\dim W_{163} = \left[ e^{\pi \sqrt{163}} \right]^2 + \left[ e^{\pi \sqrt{163}} \right] - 393768,
\]

in terms of Ramanujan’s integer

\[
\left[ e^{\pi \sqrt{163}} \right] = 262537412640768743.9999999999999925\ldots.
\]
Are linear combinations of generating functions for:

- Gauss' class numbers.
- Singular moduli on $X_0(d)$ (generate ray class fields).
- Central $L$-values of twists of elliptic curves $X_0(11), X_0(14), X_0(15), X_0(19)$.
- Central $L$-values of twists of the $X_0(31)$ abelian surface.
extraordinary MT series

are linear combinations of generating functions for:

- Gauss’ class numbers.
Can’t you just feel the moonshine?

III. New Moonshines
Pariah Moonshine

Extraordinary MT Series

Are linear combinations of generating functions for:

- Gauss’ class numbers.
- Singular moduli on $X_0(d)$ (generate ray class fields).
Extraordinary MT Series

Are linear combinations of generating functions for:

- Gauss’ class numbers.
- Singular moduli on $X_0(d)$ (generate ray class fields).
- Central $L$-values of twists of elliptic curves $X_0(11), X_0(14), X_0(15), X_0(19)$. 
Are linear combinations of generating functions for:

- Gauss’ class numbers.
- Singular moduli on $X_0(d)$ (generate ray class fields).
- Central $L$-values of twists of elliptic curves $X_0(11), X_0(14), X_0(15), X_0(19)$.
- Central $L$-values of twists of the $X_0(31)$ abelian surface.
The Module Sees Class Groups

Theorem 6 (Duncan, Mertens, O (2017))

Suppose that $-D < 0$ is a fund disc. Then the following are true:

1. If $-D < -8$ is even, then $\dim W_D \equiv -24 H(D) \pmod{2}$.
2. If $p \in \{3, 5, 7\}$, then $\dim W_D \equiv \begin{cases} -24 H(D) \pmod{3} & \text{if } p = 3, \\ -24 H(D) \pmod{p} & \text{if } p = 5, 7. \end{cases}$
The Module Sees Class Groups

Theorem 6 (Duncan, Mertens, O (2017))

Suppose that $-D < 0$ is a fund disc. Then the following are true:

1. If $-D < -8$ is even, then

   $$\dim W_D \equiv -24H(D) \pmod{2^4}.$$
Theorem 6 (Duncan, Mertens, O (2017))

Suppose that $-D < 0$ is a fund disc. Then the following are true:

1. If $-D < -8$ is even, then
   \[
   \dim W_D \equiv -24H(D) \pmod{2^4}.
   \]

2. If $p \in \{3, 5, 7\}$, \(\left(\frac{-D}{p}\right) = -1\) then
   \[
   \dim W_D \equiv \begin{cases} 
   -24H(D) \pmod{3^2} & \text{if } p = 3, \\
   -24H(D) \pmod{p} & \text{if } p = 5, 7.
   \end{cases}
   \]
The Module Sees Selmer and Tate-Shafarevich-Groups

Theorem 7 (Duncan, Mertens, O (2017))

Suppose that $E_{14} = X_0(14)$ and $E_{15} = X_0(15)$. 
The Module Sees Selmer and Tate-Shafarevich-Groups

Theorem 7 (Duncan, Mertens, O (2017))

Suppose that $E_{14} = X_0(14)$ and $E_{15} = X_0(15)$.

If $pp' = 14$ or $15$ and $\left( \frac{-D}{p} \right) = -1$ and $\left( \frac{-D}{p'} \right) = 1$, then we have:
The Module Sees Selmer and Tate-Shafarevich-Groups

Theorem 7 (Duncan, Mertens, O (2017))

Suppose that $E_{14} = X_0(14)$ and $E_{15} = X_0(15)$.

If $pp' = 14$ or $15$ and $\left( \frac{-D}{p} \right) = -1$ and $\left( \frac{-D}{p'} \right) = 1$, then we have:

1. We have that $\text{Sel}(E_N(-D))[p] \neq \{0\}$ if and only if

\[
\text{Tr}(g_{p'}|W_D) \equiv \delta_p \cdot (H(D) - \delta_p H^{(p')}(D)) \pmod{p}.
\]
The Module Sees Selmer and Tate-Shafarevich-Groups

**Theorem 7 (Duncan, Mertens, O (2017))**

*Suppose that* $E_{14} = X_0(14)$ *and* $E_{15} = X_0(15)$. If* $pp' = 14$ *or* $15 *and* $\left(\frac{-D}{p}\right) = -1$ *and* $\left(\frac{-D}{p'}\right) = 1$, *then we have:

1. We have that $Sel(E_N(-D))[p] \neq \{0\}$ if and only if

   $$\text{Tr}(g_{p'}|W_D) \equiv \delta_p \cdot (H(D) - \delta_p H^{(p')}(D)) \pmod{p}.$$

2. Suppose further that $L(E_N(-D), 1) \neq 0$. *Then* $p | \#\text{III}(E_N(-D))$ *if and only if*

   $$\text{Tr}(g_{p'}|W_D) \equiv \delta_p \cdot (H(D) - \delta_p H^{(p')}(D)) \pmod{p}.$$
Can’t you just feel the moonshine?

III. New Moonshines

Pariah Moonshine

What is Moonshine?
What is Moonshine?

Answer

Moonshine organizes special divisors on products of modular curves!
What is Moonshine?

**Answer**

Moonshine organizes special divisors on products of modular curves!

\[
V^G = \bigoplus_m V^G_m \xrightarrow{\text{moon}} (f_{[g]}) \in \bigoplus_{[g] \in \text{Conj}(G)} \text{Modular}_k(\Gamma_{[g]}) \otimes \text{Jac}(X(\Gamma_{[g]})),
\]
What is Moonshine?

**Answer**

*Moonshine organizes special divisors on products of modular curves!*

\[ V^G = \bigoplus_{m} V^G_m \xrightarrow{\text{moon}} (f_{[g]}) \in \bigoplus_{[g] \in \text{Conj}(G)} \text{Modular}_k(\Gamma_{[g]}) \otimes \text{Jac}(X(\Gamma_{[g]})). \]
What is Moonshine?

Answer

Moonshine organizes special divisors on products of modular curves!

\[ V^G = \bigoplus_m V^G_m \xrightarrow{\text{moon}} (f_{[g]}) \in \bigoplus_{[g] \in \text{Conj}(G)} \text{Modular}_k(\Gamma_{[g]}) \otimes \text{Jac}(X(\Gamma_{[g]})). \]

Evidence

- Monstrous identifies Hauptmoduln (i.e. Divisors $-\infty + a$).
What is Moonshine?

**Answer**

Moonshine organizes special divisors on products of modular curves!

\[ \mathbb{V}^G = \bigoplus_m \mathbb{V}_m^G \xrightarrow{\text{moon}} (f_{[g]}) \in \bigoplus_{[g] \in \text{Conj}(G)} \text{Modular}_k(\Gamma_{[g]}) \otimes \text{Jac}(X(\Gamma_{[g]})) . \]

**Evidence**

- Monstrous identifies Hauptmoduln (i.e. Divisors $-\infty + a$).
- Umbral cuts out individual CM (Heegner) divisors.
What is Moonshine?

Answer

Moonshine organizes special divisors on products of modular curves!

\[
V^G = \bigoplus_m V^G_m \xrightarrow{\text{moon}} (f_{[g]}), \quad [g] \in \text{Conj}(G)
\]

Evidence

- Monstrous identifies Hauptmoduln (i.e. Divisors \(-\infty + a\)).
- Umbral cuts out individual CM (Heegner) divisors.
- Pariah sums vertically and packages (all) Heegner divisors.
Our Results

Theorem 1 (Duncan, O (2016))

*The Jack Daniels problem has been resolved.*
Our Results

Theorem 1 (Duncan, O (2016))

*The Jack Daniels problem has been resolved.*

Theorem 2 (Duncan, Griffin, O (2015))

*The Monstrous Moonshine module is asymptotically regular.*

In other words, we have that

\[
\delta (m_i) = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.
\]
New Moonshines

Theorem 4 (Duncan, Griffin, O (2015))

*The Umbral Moonshine Conjecture is true.*
Can’t you just feel the moonshine?

Executive Summary

New Moonshines

Theorem 4 (Duncan, Griffin, O (2015))

The Umbral Moonshine Conjecture is true.

Theorem 5 (Duncan, Mertens, O (2017))

Moonshine exists for the O’Nan pariah sporadic group.
New Moonshines

Theorem 4 (Duncan, Griffin, O (2015))

*The Umbral Moonshine Conjecture is true.*

Theorem 5 (Duncan, Mertens, O (2017))

*Moonshine exists for the O’Nan pariah sporadic group.*

Theorems 6 and 7 (Duncan, Mertens, O (2017))

*O’Nan Moonshine informs number theory of class groups, Selmer groups and Tate-Shafarevich groups of elliptic curves.*