Lecture 1 (9/21/2018)

Recurrent Themes:

**Calculus**
- Change
  - Limit
  - Derivative
- Quantification
  - Function < Rate of change
- Integral
- Length
- Area
- Volume

**Linear Alg.**
- Structure
  - Linearity
  - Operators/Matrices
- Representation
  - Function .... Linear
- Rank
  - Determinant
  - Eigenvalues...
linear map / mapping / function / operator / transformation ...... same thing

**Ex:** \[ f(x, y) = (2x - 4y, -x - y) \]

\( f \) is a map from the plane to itself.

\[ f(0, 1) = (-4, -1) \]

\( \mathbf{u}_1 = (1, 1), \mathbf{u}_2 = (-4, 1) \) are special vectors because

\[ f(\mathbf{u}_1) = (-2, -2) = -2 \mathbf{u}_1 \]

\[ f(\mathbf{u}_2) = (-12, 3) = 3 \mathbf{u}_2 \]

Any vector in \( \mathbb{R}^2 \) is a "linear combination" of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \)

\[ \mathbf{v} = \alpha \mathbf{u}_1 + \beta \mathbf{u}_2 \] (for suitable \( \alpha \) and \( \beta \))

Then \( f \) has a simple representation through \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \):

\[ f(\mathbf{v}) = \alpha f(\mathbf{u}_1) + \beta f(\mathbf{u}_2) = -2\alpha \mathbf{u}_1 + 3\beta \mathbf{u}_2 \]
In linear algebra, the interest is not so much about computation, but representation and how structures work together.

Ex. Balance the following chemical eq.

\[ \text{Ba (OH)}_2 + \text{H}_3\text{PO}_4 \rightarrow \text{H}_2\text{O} + \text{Ba}_3(\text{PO}_4)_2 \]

In other words, find \(x, y, z, t\) (preferably integers) such that

\[ x\text{Ba (OH)}_2 + y\text{H}_3\text{PO}_4 = z\text{H}_2\text{O} + t\text{Ba}_3(\text{PO}_4)_2 \]

The number of atoms of each kind should match.

<table>
<thead>
<tr>
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<th>left</th>
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<tbody>
<tr>
<td>Ba</td>
<td>(x)</td>
<td>3t</td>
</tr>
<tr>
<td>O</td>
<td>2(x + 4y)</td>
<td>2 + 8t</td>
</tr>
<tr>
<td>H</td>
<td>2(x + 3y)</td>
<td>2z</td>
</tr>
<tr>
<td>P</td>
<td>(y)</td>
<td>2t</td>
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\[
\begin{align*}
  x - 3t &= 0 \\
  2x + 4y - 2 - 8t &= 0 \\
  2x + 3y - 2z &= 0 \\
  y - 2t &= 0
\end{align*}
\]

This is a system of eqs. of 4 eqs. and 4 unknowns.
How to solve this system?
Let's consider a simpler system: \[ \begin{align*}
5x + y &= 1 \\
2x - 3y &= 3
\end{align*} \]

There are a number of ways to solve:

1. **Elimination**:
   \[ (\text{Eq. 1}) \times 2 - (\text{Eq. 2}) : \quad 5y = -1 \quad \Rightarrow \quad y = -\frac{1}{5} \]
   Then \( x = 1 - y = \frac{6}{5} \)

2. **Substitution**:
   Eq. 1: \( x = 1 - y \)
   Eq. 2: \( 2(1-y) - 3y = 3 \quad \Rightarrow \quad y = \ldots \quad \Rightarrow \quad x = \ldots \)

3. **Geometric**: each eq. is an eq. of a line on plane. The problem g
solving the system is the problem of finding intersection.

The geometric method is not good for bigger systems of eqs. For example, it’s not clear why 4 "planes" in 4 dimensions should intersect (or usually intersect at only one point).

The first and second method are more stable under the change of sys. size. They are based on the same idea: reduce the number of unknowns and eqs. (thereby downsize the system). They are algebraic methods.

Formally, we don’t care about the unknowns $x$ and $y$, only the coefficients
That attach to them. The system is encoded in the following numbers:

\[
\begin{bmatrix}
1 & 1 & 1 \\ 2 & -3 & 3 \\
\end{bmatrix}
\]

How did we eliminate x?

\[R_2 = R_2 - 2R_1 : \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 1 \end{bmatrix}\]

Then we solve from bottom to top.

It turns out that this idea is valid/applicable for any system of linear equations. It is known as Gauss's elimination method.