Subspace of $\mathbb{R}^n$ is a subset $V \subset \mathbb{R}^n$ satisfying three following properties:

(a) $0 \in V$
(b) If $\mathbf{v}, \mathbf{w} \in V$ then $\mathbf{v} + \mathbf{w} \in V$ (closed under addition).
(c) If $\mathbf{v} \in V$ and $c \in \mathbb{R}$ then $c\mathbf{v} \in V$ (closed under scalar multiplication).

Examples:

1. $V = \{0\}$
2. $V = \{(x_1, x_2) : x_1 = 2x_2\}$ ... line passing through the origin.
3. $V = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ where $A$ is an $m \times n$ matrix.
   In this case, $V$ is called the null space of $A$.

In the second example,

\[
\begin{align*}
    x_1 - 2x_2 &= 0 \\
    \begin{bmatrix} 1 & -2 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0
\end{align*}
\]

Non examples:

1. $V = \{(x_1, x_2) : x_1 + x_2 = 1\}$ ... line not passing through the origin.

\[
\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in \mathbf{V}
\]

\[
2\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \notin \mathbf{V}
\]

2. $V = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$
   Circle is a bounded set ... can't be a subspace.

3. This subset is closed under scalar multiplication, but not under addition.

4. $\mathbb{Z} \subset \mathbb{R}^1$ is closed under addition, but not closed under scalar multiplication.
A linear combination of vectors $v_1, v_2, \ldots, v_m$ is a vector of the form $v = c_1v_1 + c_2v_2 + \ldots + c_mv_m$, where $c_1, \ldots, c_m$ are real numbers.

The set $V$ consisting of all linear combinations of vectors $v_1, v_2, \ldots, v_m$ is called the span of $v_1, v_2, \ldots, v_m$. It is the smallest subspace that contains $v_1, v_2, \ldots, v_m$. In other words, if $W$ is a subspace containing $v_1, v_2, \ldots, v_m$ then $V \subseteq W$.

Notation: $V = \text{span} \{v_1, \ldots, v_m\}$.

Example:

1. $v_1 \in \mathbb{R}^2$

   $\text{span} \{v_1\} = \{tv_1 : t \in \mathbb{R}\}$

   This is a line passing through the origin.

2. $\text{span} \{v_1, v_2\} = \begin{cases} \{0\} & \text{if } v_1 = v_2 = 0 \\ \text{line through the origin} & \text{if one vector is a multiple of the other} \\ \text{plane} & \text{if } v_1 \text{ and } v_2 \text{ are not colinear.} \end{cases}$

Vectors $v_1, v_2, \ldots, v_m$ are said to be linearly independent if $c_1v_1 + \ldots + c_mv_m \neq 0$ unless for $c_1 = \ldots = c_m = 0$.

In other words, $v_1, v_2, \ldots, v_m$ are linearly independent if there are no nontrivial linear relations among them.

Ex:

If $v_1, v_2, v_3$ satisfy $v_3 = v_1 + v_2$ then they are linearly dependent.

A nontrivial linear relation among them is $v_1 + v_2 - v_3 = 0$. 