Linear system of eqs.
\[
\begin{align*}
    x_1 + 2x_2 + x_3 &= 2 \\
    x_1 + 3x_2 &= 3 \\
    x_1 + x_2 + 2x_3 &= 1
\end{align*}
\]

Augmented matrix:
\[
\begin{bmatrix}
1 & 2 & 1 & | & 2 \\
1 & 3 & 0 & | & 3 \\
1 & 1 & 2 & | & 1
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 3 & | & 0 \\
0 & 1 & -1 & | & 1 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
\]

- \(x_3 = t\) (free variable)
- Second row: \(x_2 - x_3 = 1 \Rightarrow x_2 = 1 + t\)
- First row: \(x_1 + 3x_2 = 0 \Rightarrow x_1 = -3t\)

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
-3t \\
1 + t \\
t
\end{bmatrix}
= t
\begin{bmatrix}
-3 \\
1 \\
1
\end{bmatrix}
\]

Parametric vector form

Null space of a matrix \(A\) is defined as \(\{x : Ax = 0\}\).

This is indeed a subspace. To determine the null space is equivalent to solving a homogeneous system.

\[
\begin{bmatrix}
1 & 2 & 0 & 3 & -1 \\
-2 & -4 & 1 & -4 & 0 \\
0 & 0 & 1 & 2 & -2
\end{bmatrix}
\]

Null space of \(A\) is \(\{x \in \mathbb{R}^5 : Ax = 0\}\) (subspace of \(\mathbb{R}^5\)), denoted by \(\text{N}(A)\).

\[
A \xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 2 & 0 & 3 & -1 \\
0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
- \(x_2 = s\), \(x_4 = t\), \(x_5 = u\)
- (free variables)
From the second eq., \( x_3 = -2x_i + 2x_r = -2t + 2u \).

From the first eq., \( x_1 = -2x_i - 3x_q + x_r = -2s - 3t + u \).

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix} = \begin{bmatrix}
  -2s - 3t + u \\
  s \\
  -2t + 2u \\
  t \\
  u \\
\end{bmatrix} = s \begin{bmatrix}
  -2 \\
  1 \\
  0 \\
  0 \\
  0 \\
\end{bmatrix} + t \begin{bmatrix}
  -3 \\
  0 \\
  -2 \\
  1 \\
  0 \\
\end{bmatrix} + u \begin{bmatrix}
  1 \\
  0 \\
  0 \\
  0 \\
  1 \\
\end{bmatrix}
\]

This subspace is spanned by the vectors

\( v_1 = (-2, 1, 0, 0, 0) \)

\( v_2 = (-3, 0, -2, 0, 0) \)

\( v_3 = (1, 0, 2, 0, 1) \)

These vectors are linearly independent. As a rule, the vectors obtained by this procedure are always linearly independent (you don't need to check their linear ind. in homework or exams), thus form a basis for \( \text{N}(A) \).

The dimension of \( \text{N}(A) \) is called \textbf{nullity} of \( A \).

Observation:

\[
\text{rank}(A) = \# \text{ pivot cols in RREF of } A = \dim \text{C}(A)
\]

\[
\text{null}(A) = \dim \text{N}(A) = \# \text{ nonpivot cols of } A
\]

Thus,

\[
\text{rank}(A) + \text{null}(A) = \# \text{ cols of } A
\]

This is known as the rank-nullity theorem.