Lecture 23 (11/16/2018)

**Example:** \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) identity map

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Every direction is preserved under \( f \). All vectors are eigenvectors.

Check with computation:

* Find the eigenvalues of \( A \):

\[
\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2
\]

This polynomial has double root \( \lambda_1 = \lambda_2 = 1 \).

* Find eigenvectors of \( \lambda_1 \):

\[
A - \lambda_1 I = A - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

The null space = \( \mathbb{R}^2 \).

\[ E(\lambda_1) = \mathbb{R}^2 = \text{span} \{ e_1, e_2 \} \]

The matrix (or the linear map) is diagonalizable.

**Example:**

\[ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad (f: \mathbb{R}^2 \to \mathbb{R}^2, f(2y) = (x + 2y, y)) \]

**Eigenvalues:**

\[
\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 \quad \text{double root} \quad \lambda_1 = \lambda_2 = 1
\]

**General rule:**

If \( A \) is a upper (or lower) triangular matrix then the eigenvalues of \( A \) are the entries on the diagonal.

**Eigenvalues:**

\[ A - \lambda I = A - I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{row} \begin{bmatrix} 0 & 1 \end{bmatrix} \]

\[ x_1 = t_1, x_2 = 0 \]
E(\lambda) = \{ [t] : t \in \mathbb{R} \} = \text{span} \{ [t] \}

*the set of eigenvectors corresponding to \lambda

We need 2 linearly independent vectors to form a basis of \( \mathbb{R}^2 \). Here we have only one. Thus, there is no basis of \( \mathbb{R}^2 \) that contains only eigenvectors. \( A \) (or \( f \)) is not diagonalizable.

To diagonalize a matrix \( A \) is to find matrices \( P \) and \( D \) such that

\[
P \text{ is invertible, } D \text{ is diagonal, } \quad D = P^{-1}AP.
\]

In terms of linear maps, to diagonalize a linear map \( f \) is to find a basis \( S = \{ v_1, v_2, \ldots, v_n \} \) such that \( f \) acts on each direction \( v_1, v_2, \ldots, v_n \) as scalings. The eigenvalues are the scaling factors.

The connection between two pictures: \( P \) is the matrix whose columns are vectors in \( S \). And \( D \) is a diagonal matrix whose entries on the diagonal are scaling factors.

Suppose we have a polynomial \( Q(\lambda) = (\lambda-1)^4(\lambda-2)^3 \). This polynomial has only two distinct roots: \( \lambda = 1 \) with multiplicity 4, \( \lambda = 2 \) with multiplicity 3.

**General rule:**

The number of linearly independent eigenvectors corresponding to an eigenvalue does not exceed the multiplicity of the eigenvalue in the characteristic polynomial.

For example, suppose the matrix \( Q \) above is the characteristic of a 7x7 matrix (or a linear map \( f : \mathbb{R}^7 \to \mathbb{R}^7 \)). Then \( E(1) \) has dimension \( \leq 4 \), and \( E(2) \) has dimension \( \leq 3 \).
Theorem: If the dimension of the eigenspace \( E(\lambda) \) is equal to the multiplicity of \( \lambda \) for each eigenvalue \( \lambda \), then the given matrix is diagonalizable. Otherwise, it is not diagonalizable.

*Procedure to diagonalize a matrix \( A \) (size \( n \times n \))*

1) **Compute the eigenvalues:**

   - If the matrix is already in upper (or lower) triangular form, then the eigenvalues are the entries on the diagonal. For example,
     \[
     A = \begin{bmatrix}
     1 & 2 & 4 & 1 \\
     0 & 1 & -1 & 2 \\
     0 & 0 & 2 & 3 \\
     0 & 0 & 0 & 1
     \end{bmatrix}
     \]
     then the eigenvalues of \( A \) are \( \lambda_1 = 1 \) (with multiplicity 3) and \( \lambda_2 = 2 \) (with multiplicity 1).

   - If the matrix is not in upper (or lower) triangular form, then compute the characteristic polynomial:
     \[
     q(\lambda) = \det(A - \lambda I).
     \]
     Factor this polynomial to find roots (with multiplicity).

2) To each eigenvalue \( \lambda \), find the corresponding eigenvectors.

   - This is to find the null space of \( A - \lambda I \).
   - Select a basis for \( E(\lambda) \).

   - If the dimension of \( E(\lambda) \) is less than the multiplicity of \( \lambda \), then STOP: the matrix is not diagonalizable.

3) Put together the basis of \( E(\lambda) \) to obtain the matrix \( P \).
4) Matrix \( D \) is determined by the eigenvalues corresponding to the columns of \( P \).

*We will consider a few examples next time.*