Some review problems for Final

In the following problems, verify your answer with valid arguments. Make sure to write in full sentences.

1. Let $V = P_2(\mathbb{R})$ be the vector space of all polynomials of degree $\leq 2$ with real coefficients. Let

$$V_1 = \{ f \in V : f(1) = 0 \},$$
$$V_2 = \{ f \in V : f(2) = 0 \}.$$

Is $V_1 + V_2$ a direct sum?

2. Let $V = P_2(\mathbb{R})$. Define $\phi(u) = |u(1)| + |u(2)|$ for any $u \in V$. Is $\phi$ a norm on $V$?

3. Let $V = P_2(\mathbb{R})$. Define $\phi(u) = |u(1)| + |u(2)| + |u(3)|$ for any $u \in V$. Show that $\phi$ is a norm on $V$.

4. Put

$$V_1 = \{ A \in M_{2 \times 2}(\mathbb{R}) : A = A^T \}$$
$$V_2 = \{ A \in M_{2 \times 2}(\mathbb{R}) : A = -A^T \}.$$

Show that $V_1 \oplus V_2 = M_{2 \times 2}$.

5. Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_2 = x_1 + x_3, x_3 = 2x_1 - x_2 + 5x_4\}$. Find a subspace $W$ of $\mathbb{R}^4$ such that $V \oplus W = \mathbb{R}^4$.

6. Let $V$ be the vector space of all smooth functions from $\mathbb{R}$ to itself. Let $F : V \to V$ be a linear map defined by $F(u) = u' - u$. Let $W$ be the vector space of all smooth functions satisfying the differential equation $u'' + u' + u = 0$. Show that $W$ is invariant under $F$.

7. Let $V = M_{2 \times 2}(\mathbb{R})$. Let $f : V \to V$ be a linear map defined by $f(A) = A^T$. Is $f$ diagonalizable? If it is, find a basis of $V$ in which $f$ is represented by a diagonal matrix.