In the following problems, make sure to write your arguments coherently in full sentences. If possible, start a sentence with words rather than a formula. Avoid using ambiguous symbols such as →, ?, . . . , ∴. Instead, use words to transition your ideas, for example “This leads to”, “Therefore”, “We want to show”, etc.

1. Let $V$ be a vector space over a field of numbers $F$ (which could be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). Let $U$ be a subspace of $V$. Show that $U + U = U$. Under what condition of $U$ is this sum a direct sum?

2. Let $U = \{(x, y, y, x) : x, y \in \mathbb{R}\}$. This is a subspace of $\mathbb{R}^4$. Find a subspace $V$ of $\mathbb{R}^4$ such that $U \oplus V = \mathbb{R}^4$.

3. Consider two vector spaces $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, a + d = b + c = 0 \right\}$, $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, a = d = 0 \right\}$.

   (a) Find a basis of $U + V$.
   
   *Hint:* You can use Matlab to compute RREF of a matrix. (Make sure to write the Matlab code on your homework. See lecture note on 11/08 for an example.) If you choose not to use Matlab, make sure that you write all row reduction steps.

   (b) Is $U + V$ a direct sum?

Do the following problem for 6 bonus points.

4. Let $V = \{ z \in \mathbb{C} : z(1 + i) + 2\bar{z} = 0 \}$. Is $V$ a vector space over $\mathbb{C}$ ?