Homework 5
Answer Key

Consider the following subspaces of \( \mathbb{R}^3 \):

\[ U = \{ (x_1, x_2, x_3) : x_1 = x_2 + x_3 \} \]
\[ V = \{ (x_1, x_2, x_3) : x_1 = x_2 \} \]
\[ W = \{ (x_1, x_2, x_3) : x_1 = x_2 = x_3 \} \]

1. Find a basis of the intersection \( U \cap V \). What is the dimension?

**Solution:** We can write the intersection as a subset of \( \mathbb{R}^3 \) by including the conditions for both \( U \) and \( V \):

\[ U \cap V = \{ (x_1, x_2, x_3) : x_1 = x_2 + x_3, \ x_1 = x_2 \} \]
\[ = \{ (x_1, x_1, x_1) : x_1 = x_1 + x_1 \} \]
\[ = \{ (x_1, 0, 0) : x_1 \in \mathbb{R} \} \]
\[ = \{ x_1 (1, 1, 0) : x_1 \in \mathbb{R} \} \]

The set \( B = \{(1,1,0)\} \) spans \( U \cap V \), and since it is a set of only one non-zero element, \( B \) is linearly independent. Therefore \( B \) is a basis for \( U \cap V \) and the dimension is

\[ \dim(U \cap V) = 1. \]

2. Find a basis of \( U \cap W \). What is the dimension?

**Solution:** We can write the intersection as a subset of \( \mathbb{R}^3 \) by including the conditions for both \( U \) and \( W \):

\[ U \cap W = \{ (x_1, x_2, x_3) : x_1 = x_2 + x_3, \ x_1 = x_2 = x_3 \} \]
\[ = \{ (x_1, x_1, x_1) : x_1 = x_1 + x_1 \} \]
\[ = \{ (0,0,0) \} \]

Therefore \( U \cap W \) is the vector space of only the zero element, so a basis for \( U \cap W \) is the empty set \( B = \emptyset \). The dimension of \( U \cap W \) is the size of the empty set:

\[ \dim(U \cap V) = 0. \]
3. Show that $U + W = \mathbb{R}^3$.

**Solution:** We want to show that $U + W$ contains a basis for $\mathbb{R}^3$. We will start by finding bases for $U$ and $W$ separately.

$$U = \{(x_1, x_2, x_3) : x_1 = x_2 + x_3\}$$
$$= \{(x_2 + x_3, x_2, x_3) : x_2, x_3 \in \mathbb{R}\}$$
$$= \{x_2(1, 1, 0) + x_3(1, 0, 1) : x_2, x_3 \in \mathbb{R}\}$$

and a basis for $U$ is $\{(1, 1, 0), (1, 0, 1)\}$.

$$W = \{(x_1, x_2, x_3) : x_1 = x_2 = x_3\}$$
$$= \{x_1(1, 1, 1) : x_1 \in \mathbb{R}\}$$

and a basis for $W$ is $\{(1, 1, 1)\}$. The union of these two bases is

$$B = \{(1, 1, 0), (1, 0, 1), (1, 1, 1)\}.$$ 

Since $B$ is the union of a basis for $U$ and a basis for $W$, that means $\text{span}(B) = U + W$.

We want to check that $B$ is linearly independent (and hence a basis for $\mathbb{R}^3$). Consider the equation

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ 

Writing this as an augmented matrix and using row reduction gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ 

Therefore $c_1 = c_2 = c_3 = 0$, so $B$ is linearly independent. Since $\mathbb{R}^3$ has dimension 3 and $B$ is a linearly independent subset of $\mathbb{R}^3$ with 3 elements, $B$ must be a basis for $\mathbb{R}^3$. Therefore

$$\mathbb{R}^3 = \text{span}(B) = U + W.$$ 

4. Show that $V + W = V$.

**Solution:** The only way for this to be true is if $W$ is a subset of $V$. Notice that any element $(x_1, x_2, x_3) \in \mathbb{R}^3$ in $W$ satisfies $x_1 = x_2$. This is the only condition required for $V$, so any element of $W$ is also an element of $V$.

If $v \in V$ and $w \in W$, then $v + w \in V$ since $w \in W \subseteq V$. Therefore

$$V + W = \{v + w : v \in V, w \in W\} \subseteq V.$$ 

You saw in class that $V$ is always a subset of $V + W$ (just let $w = 0$). Therefore $V + W = V$. 
