1. Let \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a linear map given by \( f(v) = Av \) where

\[
A = \begin{bmatrix}
3 & -2 & -2 \\
-1 & 4 & 2 \\
2 & -4 & -2
\end{bmatrix}.
\]

Is \( f \) diagonalizable? If it is, express \( V = \mathbb{R}^3 \) as a direct sum of one-dimensional invariant subspaces under \( f \); then find a basis of \( V \) in which \( f \) is represented by a diagonal matrix.

**Solution:** First find the characteristic polynomial of \( A \):

\[
\text{det}(A - \lambda I) = \begin{vmatrix}
3 - \lambda & -2 & -2 \\
-1 & 4 - \lambda & 2 \\
2 & -4 & -2 - \lambda
\end{vmatrix}
\]

\[
= (3 - \lambda)[(4 - \lambda)(-2 - \lambda) - (2)(-4)]
+ [(-2)(-2 - \lambda) - (-2)(-4)]
+ (2)[(-2)(-2) - (-2)(4 - \lambda)]
\]

\[
= (3 - \lambda)(\lambda^2 - 2\lambda) + (2\lambda - 4) + (2)(-2\lambda + 4)
\]

\[
= (3 - \lambda)(\lambda)(\lambda - 2) + 2(\lambda - 2) - 4(\lambda - 2)
\]

\[
= (\lambda - 2)(3 - \lambda)(\lambda + 2 - 4)
\]

\[
= (\lambda - 2)(\lambda^2 - 3\lambda + 2)
\]

\[
= -\lambda^3 + 2\lambda^2 + \lambda - 2
\]

Setting \(-\lambda^2 + 2\lambda - 1 = 0\) gives eigenvalues \( \lambda = 1 \) and \( \lambda = 2 \).

To see if \( f \) is diagonalizable, we need to check if the direct sum of the eigenspaces is all of \( V \). That is, we need to check if the dimensions of the eigenspaces add up to \( \dim(V) = 4 \).

We now need to find the eigenvectors of \( A \). This means solving the equation

\[(A - \lambda I)v = 0\]

where \( v = (v_1, v_2, v_3) \in V = \mathbb{R}^3 \).

First let \( \lambda = 1 \). The augmented form of the above equation is

\[
\begin{bmatrix}
3 - 1 & -2 & -2 & 0 \\
-1 & 4 - 1 & 2 & 0 \\
2 & -4 & -2 - 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & -2 & -2 & 0 \\
-1 & 3 & 2 & 0 \\
2 & -4 & -3 & 0
\end{bmatrix}.
\]

(continued on next page)
Now use row reduction:

\[
\begin{bmatrix}
2 & -2 & -2 & 0 \\
-1 & 3 & 2 & 0 \\
2 & -4 & -3 & 0
\end{bmatrix}
\xrightarrow{R_1 \rightarrow R_1}
\begin{bmatrix}
1 & -1 & -1 & 0 \\
-1 & 3 & 2 & 0 \\
2 & -4 & -3 & 0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & -1 & -1 \\
0 & 2 & 1 \\
0 & -2 & -1
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 + R_1}
\begin{bmatrix}
1 & 1 & 0 \\
0 & 2 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

This gives the equations

\[
v_1 + v_2 = 0 \\
2v_2 + v_3 = 0
\]

so \(v_1 = -v_2\) and \(v_3 = -2v_2\). The eigenvectors \(v\) can be written as

\[
v = \begin{bmatrix}
-v_2 \\
v_2 \\
-2v_2
\end{bmatrix} = v_2 \begin{bmatrix}
-1 \\
1 \\
-2
\end{bmatrix}
\]

where \(v_2 \in \mathbb{R}\). Therefore, a basis for the eigenspace \(E_1\) is

\[
B_1 = \{(-1, 1, -2)\}
\]

Now let \(\lambda = 2\). We follow the same process of solving \((A - \lambda I)v = 0:\)

\[
\begin{bmatrix}
3 & -2 & -2 & 0 \\
-1 & 4 & -2 & 2 \\
2 & -4 & -2 & 2
\end{bmatrix}
\xrightarrow{\text{row reduction}}
\begin{bmatrix}
1 & -2 & -2 \\
-1 & 2 & 2 \\
2 & -4 & -4
\end{bmatrix}
\]

so we get the single equation

\[
v_1 - 2v_2 - 2v_3 = 0.
\]

This means that \(v_1 = 2v_2 + 2v_3\), so the eigenvectors \(v\) can be written as

\[
v = \begin{bmatrix}
2v_2 + 2v_3 \\
v_2 \\
v_3
\end{bmatrix} = v_2 \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} + v_3 \begin{bmatrix}
2 \\
0 \\
1
\end{bmatrix}
\]

where \(v_2, v_3 \in \mathbb{R}\). A basis for \(E_2\) is

\[
B_2 = \{(2, 1, 0), (2, 0, 1)\}.
\]

Since \(\dim(E_1) + \dim(E_2) = 1 + 2 = 3 = \dim(V)\) the linear map \(f\) is diagonalizable. Under the basis

\[
B = B_1 \cup B_2 = \{(-1, 1, 2), (2, 1, 0), (2, 0, 1)\}
\]

the map \(f\) is represented by the diagonal matrix

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]
2. Let \( f: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}) \) be a linear map given by

\[
  f \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & 2a + b \\ 2c & c + 2d \end{bmatrix}.
\]

Is \( f \) diagonalizable? If it is, express \( V = M_{2\times 2}(\mathbb{R}) \) as a direct sum of one-dimensional invariant subspaces under \( f \); then find a basis of \( V \) in which \( f \) is represented by a diagonal matrix.

**Solution:** Since \( f \) is a map from \( M_{2\times 2}(\mathbb{R}) \) to \( M_{2\times 2}(\mathbb{R}) \) we begin by finding a basis for \( M_{2\times 2}(\mathbb{R}) \) and representing \( f \) as a \( 4 \times 4 \) matrix. Consider the basis \( B \) for \( M_{2\times 2}(\mathbb{R}) \) given by

\[
  B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

Applying \( f \) to the first element of \( B \), we get

\[
  f \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\]

so the first row of the matrix for \( f \) is

\[
  \begin{bmatrix} f \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \end{bmatrix}_{B,B} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.
\]

Continuing in this way, we find that

\[
  A = [f]_{B,B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.
\]

Since this is a triangular matrix, the eigenvalues are simply the entries on the diagonal. That is, the eigenvalues are \( \lambda = 1 \) and \( \lambda = 2 \).

To see if \( f \) is diagonalizable, we need to check if the direct sum of the eigenspaces is all of \( V \). That is, we need to check if the dimensions of the eigenspaces add up to \( \dim(V) = 4 \).

We must calculate the eigenspace for each eigenvector. We want to solve

\[(A - \lambda I)v = 0\]

for \( v = (v_1, v_2, v_3, v_4) \in V \). First, let \( \lambda = 1 \). The augmented form of the above equation is

\[
  \begin{bmatrix} 1-1 & 0 & 0 & 0 & 0 \\ 2 & 1-1 & 0 & 0 & 0 \\ 0 & 0 & 2-1 & 0 & 0 \\ 0 & 0 & 1 & 2-1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \overset{\text{row reduction}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

so \( v_1 = v_3 = v_4 = 0 \) and the eigenvectors \( v \) can be written as

\[
  v = \begin{bmatrix} v_1 \\ v_2 \\ 0 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\]

where \( v_2 \in \mathbb{R} \). A basis for the eigenspace \( E_1 \) is \( B_1 = \{(0,1,0,0)\} \) so \( \dim(E_1) = 1 \).

(continued on next page)
Now let $\lambda = 2$. We follow the same process of solving $(A - \lambda I)v = 0$:

$$
\begin{bmatrix}
1 - 2 & 0 & 0 & 0 \\
2 & 1 - 2 & 0 & 0 \\
0 & 0 & 2 - 2 & 0 \\
0 & 0 & 1 & 2 - 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
$$

so $v_1 = v_2 = v_3 = 0$ and the eigenvector $v$ can be written as

$$
v = \begin{bmatrix}
0 \\
0 \\
0 \\
v_4
\end{bmatrix} = v_4
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
$$

where $v_4 \in \mathbb{R}$. A basis for the eigenspace $E_2$ is $B_2 = \{(0, 0, 0, 1)\}$ so $\dim(E_2) = 1$.

Since $\dim(E_1) + \dim(E_2) = 1 + 1 = 2 \neq 4 = \dim(V)$
the linear map $f$ is not diagonalizable.

3. Let $V$ be the vector space of all smooth (infinitely differentiable) functions from $\mathbb{R}$ to $\mathbb{R}$. Let $F: V \rightarrow V$ be a linear map defined by $F(u) = u'$. Find all eigenvectors and eigenvalues of $F$.

Solution: Since $V$ is an infinite-dimensional vector space, we cannot solve this problem by converting to a matrix equation. Instead, we need to find the eigenvectors directly. Recall that an eigenvector of $F$ is a vector $v \in V$ such that

$$
F(v) = \lambda v
$$

for some $\lambda$ in the base field $F = \mathbb{R}$. In this case, we get the equation

$$
v' = \lambda v,
$$

where $v = v(t)$ is a (smooth) function of the variable $t$. This is a differential equation that can be solved using separation of variables:

$$
v' = \lambda v, \\
\int \frac{1}{v} v' dt = \int \lambda dt \\
\ln(v) = \lambda t + C \\
v = e^{\lambda t + C} = De^{\lambda t}
$$

In this case there was no restriction on $\lambda$ when solving for $v$ above, so every element of $\mathbb{R}$ is an eigenvalue of $F$. The eigenvectors associated to the eigenvalue $\lambda$ are

$$
\{De^{\lambda t} : D \in \mathbb{R}\}
$$

A basis for the eigenspace $E_\lambda$ is $B_\lambda = \{e^{\lambda t}\}$, so each eigenspace has dimension 1.
4. Let \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) be a linear map given by \( f(v) = Av \) where

\[
A = \begin{bmatrix}
1 - i & 2 - i \\
0 & 2 + i
\end{bmatrix}.
\]

Is \( f \) diagonalizable? If it is, express \( V = \mathbb{C}^2 \) as a direct sum of one-dimensional invariant subspaces under \( f \); then find a basis of \( V \) in which \( f \) is represented by a diagonal matrix.

**Solution:** Notice that \( A \) is a triangular matrix, so the eigenvalues are the entries on the diagonal:

\[
\lambda = 1 - i \quad \text{or} \quad \lambda = 2 + i.
\]

Since \( V \) has dimension 2 and there are 2 distinct eigenvalues, we already know that \( f \) is diagonalizable. We now need to find the eigenspace for each eigenvalue. That is, we need to solve

\[
(A - \lambda I)v = 0
\]

for \( v = (v_1, v_2) \in V = \mathbb{C}^2 \). First let \( \lambda = 1 - i \):

\[
\begin{bmatrix}
1 - i - (1 - i) & 2 - i \\
0 & 2 + i - (1 - i)
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
0 & 2 - i \\
0 & 1 + 2i
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}.
\]

Perform row reduction:

\[
\begin{bmatrix}
0 & 2 - i \\
0 & 1 + 2i
\end{bmatrix}
\begin{bmatrix}
1 - 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 - i \\
0 & 1 + 2i
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}.
\]

This gives \( v_2 = 0 \), so the eigenvectors \( v \) are

\[
v = \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = v_1 \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

where \( v_1 \in \mathbb{C} \). A basis for the eigenspace \( E_{1-i} \) is \( \{(1, 0)\} \).

Now let \( \lambda = 2 \). We follow the same process of solving \( (A - \lambda I)v = 0 \):

\[
\begin{bmatrix}
1 - i - (2 + i) & 2 - i \\
0 & 2 + i - (2 + i)
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
-1 - 2i & 2 - i \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}.
\]

Perform row reduction:

\[
\begin{bmatrix}
-1 - 2i & 2 - i \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 - 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 2 - i \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}.
\]

(continued on next page)
The fraction in the reduced matrix can be simplified:

\[
\frac{2 - i}{-1 - 2i} = \frac{2 - i}{-1 - 2i} \cdot \frac{-1 + 2i}{-1 + 2i} = \frac{(2 - i)(-1 + 2i)}{(-1)^2 - (2i)^2} = \frac{-2 + 4i + i + 2}{1 + 4} = \frac{5i}{5} = i,
\]

so the reduced matrix is

\[
\begin{bmatrix}
1 & i & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

This gives \( v_1 + iv_2 = 0 \), so \( v_1 = -iv_2 \). The eigenvectors \( v \) can be written as

\[
v = \begin{bmatrix} -iv_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}
\]

where \( v_2 \in \mathbb{C} \). A basis for the eigenspace \( E_{2+i} \) is \( \{(i, 1)\} \).

As we noted earlier, \( f \) is diagonalizable. Under the basis

\[
B = \{(1, 0), (-i, 1)\}
\]

the map \( f \) is represented by the diagonal matrix

\[
\begin{bmatrix}
1 - i & 0 \\
0 & 2 + i
\end{bmatrix}.
\]