\[ \phi: V \rightarrow W \text{ linear map} \]
\[ V \text{ basis } B_1 = \{ v_1, v_2, \ldots, v_m \} \]
\[ W \text{ basis } B_2 = \{ w_1, w_2, \ldots, w_m \} \]

The matrix representation of \( \phi \) in these bases is
\[
[\phi]_{B_2, B_1} = \begin{bmatrix} 
[\phi(v_1)]_{B_2} & \cdots & [\phi(v_m)]_{B_2} 
\end{bmatrix}
\]

Important property: for any \( v \in B_1 \)
\[
[\phi(v)]_{B_2} = [\phi]_{B_2, B_1} [v]_{B_1}
\]
\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]
\[
mx1 \quad mxn \quad nx1
\]

With this property, one can work on linear maps on general vector spaces through coordinate vectors.

Operations on linear maps

Let \( \phi \) and \( \psi \) be linear maps from \( V \) to \( W \) (over a field \( F \)). Then \( \phi \circ \psi \in W^V \) (the vector space of all functions from \( V \) to \( W \)).

We already know how to add \( \phi \) and \( \psi \) and how to scale each map:

\[
(\phi + \psi)(v) = \phi(v) + \psi(v) \quad \forall v \in V
\]
\[
(c \phi)(v) = c \phi(v) \quad \forall v \in V, \forall c \in F
\]

In coordinates:
\[
[\phi + \psi]_{B_2, B_1} = [\phi]_{B_2, B_1} + [\psi]_{B_2, B_1} \quad \text{(addition of matrices)}
\]
\[
[c \phi]_{B_2, B_1} = c [\phi]_{B_2, B_1} \quad \text{(scaling a matrix)}
\]

What is the operation of linear maps that corresponds to matrix multiplication? Answer: map composition.
Consider two linear maps:

$$V_0 \xrightarrow{f} V_1 \xrightarrow{g} V_2$$

Let $B_0, B_1, B_2$ be bases of $V_2, V_1, V_2$ respectively. It is good exercise to show that the composite map $g \circ f$ is also a linear map. Let us assume this property.

Because $g \circ f$ is a linear map from $V_2$ to $V_2$, the matrix representation of $g \circ f$ in basis $B_0$ and $B_2$ is $[g \circ f]_{B_2, B_0}^{B_0, B_2}$. We have an interesting property as follows:

$$[g \circ f]_{B_2, B_0}^{B_0, B_2} = [g]_{B_2, B_1}^{B_1, B_0} [f]_{B_1, B_0}^{B_0, B_2}$$

matrix multiplication

Note that the rule of multiplication of matrices was defined such that matrix multiplication is compatible with composition of linear maps. We observe that operations on linear maps are compatible with operations on their representation matrices.

In Math 341, we learned null space and range space of a matrix: if $A$ is an $m \times n$ matrix, then

$$\text{null}(A) = \{ v \in \mathbb{R}^n : A v = 0 \}$$

$$\text{range}(A) = \{ A v : v \in \mathbb{R}^n \}$$

In some textbooks, null $(A)$ is denoted as $\ker(A)$ (kernel), range $(A)$ is denoted as $\text{im}(A)$ (image).

The corresponding definitions in linear maps are:

$$f : V \rightarrow W \text{ linear}$$

$$\text{null}(f) = \{ v \in V : f(v) = 0 \}$$

$$\text{range}(f) = \{ f(v) : v \in V \}$$

We will consider some examples next time.