f: V → W linear

If \( \dim V > \dim W \) then \( f \) is not monomorphic.

"Large blanket \( V \) has to be folded to put in the box \( W \)"

If \( \dim V < \dim W \) then \( f \) is not epimorphic.

"\( W \) is too big to be covered by \( V \)."

Theorem: If \( \dim V = \dim W < \infty \), then monomorphism, epimorphism, isomorphism are equivalent.

Why so?

Suppose \( f: V \to W \) is a monomorphism. By rank-nullity theorem,

\[
\dim \text{null}(f) + \dim \text{range}(f) = \dim V = \dim W
\]

\( = 0 \) since \( \dim V = \dim W \)

\( f \) is monomorphic

Then \( \dim \text{range}(f) = \dim W \). Because \( \text{range}(f) \) is a subspace of \( W \) and has the same dimension as \( W \), it must be equal to \( W \). Hence, \( f \) is epimorphic.

Next, we give discuss some quick ways to check if a linear map \( f: V \to W \) is monomorphic/epimorphic/isomorphich.

- Check if \( f \) is monomorphic:
  
  If \( \dim V > \dim W \) then conclude that \( f \) is not monomorphic.
  
  Otherwise, one can attempt to show that \( f \) is monomorphic by...
the following methods:

1) Use definition, i.e. show that \( \text{null}(f) = \{0\} \).
   For this method, one can start the proof by saying: "let \( v \in V \) such that \( f(v) = 0 \). We want to show \( v = 0 \)."

2) Check if \( \dim \text{range}(f) = \dim V < \infty \).
   Once this is shown, \( \text{null}(f) = \{0\} \) because of rank-nullity theorem:
   \[
   \frac{\dim \text{null}(f) + \dim \text{range}(f)}{\dim V} = 1
   \]
   For this method, one can start by writing
   \[
   \text{range}(f) = \{ \ldots \} = \text{span} \{\ldots\} \quad \text{(try to find a spanning set)}
   \]
   Then try to find a basis of \( \text{range}(f) \). It is usually this spanning set (but one needs to check if the spanning set is linearly independent).
   Then one compares the dimension of \( \text{range}(f) \) with dimension of \( V \).

- Check if \( f \) is epimorphic:
  If \( \dim V < \dim W \) then conclude that \( f \) is not epimorphic.
  Otherwise, one can attempt to show that \( f \) is epimorphic by the following methods:

1) Use definition, i.e. show that \( \text{range}(f) = W \).
   Since \( \text{range}(f) \) is a subspace of \( W \), it suffices to show that \( \dim \text{range}(f) = \dim W \).
   For this method, one can start by writing
   \[
   \text{range}(f) = \{ \ldots \} = \text{span} \{\ldots\} \quad \text{(try to find a spanning set)}
   \]
   Then try to find a basis of \( \text{range}(f) \). It is usually this spanning set (but one needs to check if the spanning set is linearly independent).
   Then one compares the dimension of \( \text{range}(f) \) with dimension of \( W \).
2) Check if \( f \) is onto.

For this method, one can start by writing: "Let \( w \in W \). We want to find \( v \in V \) such that \( f(v) = w \)."

- Check if \( f \) is isomorphic:

If \( \dim V \neq \dim W \) then conclude that \( f \) is not isomorphic. Otherwise, one can attempt to show that \( f \) is isomorphic by showing that \( f \) is monomorphic (or epimorphic). Note that in this case (\( \dim V = \dim W \)), isomorphism is equivalent to monomorphism and is equivalent to epimorphism.

One can start, for example, by writing that: "Let \( v \in V \) such that \( f(v) = 0 \). We want to show \( v = 0 \)."

See examples on the worksheets.