Vector space $\mathbb{R}^n$: one can talk about addition of two vectors, scaling of a vector, linear combination, linear independence, spanning set (or generating set), basis.

To study these topics, one only focuses on a few features of $\mathbb{R}^n$: being able to add and scale vectors. One doesn’t need to care about how to multiply or divide two vectors. In fact, with $V = \mathbb{R}^n$ and $F = \mathbb{R}$ we observe that the following properties are satisfied.

(A) Addition:

(A0) There is an "intrinsic" addition operator:

If $x, y \in V$ then $x + y \in V$.

(A1) $x + y = y + x$ (commutative)

(A2) $(x + y) + z = x + (y + z)$ (associative)

(A3) $x + "0" = x$ (having an additive identity)

(A4) $x + "-x" = 0$ (having an additive inverse)

(S) Scaling (or scalar multiplication) by factors in $F$:

(S0) There is an "external" multiplication with numbers:

If $x \in V$ and $c \in F$ then $cx \in V$.

(S1) $c(dx) = (cd)x$ (associative)

scaling \hspace{1cm} product \hspace{1cm} scaling

of number

(S2) $1x = x$ (scaling by 1)

The set of numbers can be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (called field).

(I) Interaction between adding and scaling:

(I1) $c(x+y) = cx + cy$ ("first add, then scale" is the same as "first scale, then add")

(I2) $(c+d)x = cx + dx$

These two identities are called distribution property.
A structure consisting of a set $V$, a set of numbers $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, addition, scalar multiplication that satisfies (A), (S), (I) is called a vector space over $F$.

Each of properties (A0), (A1), ..., (I2) is called an axiom.

One can think of "vector space" as a recipe in a cookbook. A recipe has two parts: ingredients and directions.

**Ingredients:**
- set $V$
- field of numbers $F$
- addition
- scaling

**Directions:** (A), (S), (I) are satisfied.

**Ex 1:** Only assuming (A0), (A1), (A2), is it true that 
\[(u+w)+w = \pi + (u+w) \text{ for all } u, v, w \in V ?\]

Yes. The reason is as follows.
\[(u+w)+w = (\pi+u)+w \text{ by (A1)}
= \pi + (u+w) \text{ by (A2)}\]

**Ex 2:** Only assuming group (A), is it true that each $v \in V$ has exactly one additive inverse?

Yes. Let $v \in V$. Suppose $u$ and $w$ are inverses of $v$. We want to show that $u = w$.

The only thing we know about $u$ and $w$ is

\[u+v = 0\]
\[w+v = 0\]

Add both sides of the first equation to $w$:

\[(u+v) + w = 0 + w\]
By (A3), \( \text{RHS} = w \)
By (A2), \( \text{LHS} = u + (v + w) \)
\[ = u + 0 \]
\[ = u \text{ by (A3)} \]
Therefore, \( u = w \)

Ex3: Assuming only (A) and (S), is it true that \((-1)v = -v\) for all \(v \in V\)?

Well, let’s analyse. The equation

\[-(1)v = -v \quad (1)\]

is equivalent to

\[-(1)v + v = (-v) + v \quad (2)\]

In other words, if (1) is true then (2) is true. If (2) is true, then (1) is true (by adding \(-v\) on both sides).

We see that \( \text{RHS} (2) = 0 \) by (A4).

\[\text{LHS} (2) = -(1)v + 1v\]

We cannot write

\[-(1)v + v = (-1 + 1)v\]

because this would need (I). The property (II) is included in the definition of vector space so that the identity (I) is true.

*Comments:*

- If \( V = \mathbb{R}^n \) and \( F = \mathbb{R} \), the answer to Ex1,2,3 is obviously yes.

However, for abstract \( V \), the answer is not obvious and requires proof as we showed above.

- Why do we need to care about such an abstract definition? Should we just be content with \( \mathbb{R}^n \)?
Last time we considered an example in which the idea of eigenvectors yields an important result in mathematics and signal processing—Fourier series. Each “vector” was a function (a much more complicated object than a vector in \( \mathbb{R}^n \)).

The abstract notion of vector space makes \( \mathbb{R}^n \) more useful; many results on \( \mathbb{R}^n \) can be used for other sets, for example the set of functions or matrices.