* There are several ways to check if $U + V$ is a direct sum.

Below are some (usually) quick ways:

1) Check if $U \cap V = \{0\}$. For this method, one can start by writing: "Let $z \in U \cap V$. We check if (or show that) $v = 0$.

2) Find a basis $B_1$ of $U$, basis $B_2$ of $V$. Then check if $B_1 \cup B_2$ is linearly independent. Note that concatenation is almost the same as union, except that we don't remove the repeated vector. For example,

\[ B_1 = \{(0,1,0), (0,0,1)\} \]
\[ B_2 = \{(1,0,0), (0,0,1)\} \]

Then

\[ B_1 \cup B_2 = \{(0,1,0), (0,0,1), (1,0,0)\} \text{ (union)} \]
\[ B_1 \cup B_2 = \{(0,1,0), (0,0,1), (1,0,0)\} \text{ (concatenation)} \]

In this example, $B_1 \cup B_2$ is linearly independent, but $B_1 \cup B_2$ is not.

3) If $U, V \subset W$, and $\dim U + \dim V > \dim W$ then $U + V$ is not a direct sum.

Why?

We know that $U + V$ is the smallest vector space that contains both $U$ and $V$. Since $W$ contains both $U$ and $V$, it must contain $U + V$. Then

\[ \dim (U + V) \leq \dim W < \dim U + \dim V. \]

Thus, $U + V$ is not a direct sum.

**Ex:** Consider two subspaces of $\mathbb{R}^3$:
$U = \{ (x,y,z) : x+y = 0 \}$
$V = \{ (x,y,z) : z+y = 0 \}$

Show that $U + V$ is not a direct sum.

One can find the intersection $U \cap V$ can realize that it is not $\{0\}$. Geometrically, $U$ and $V$ are planes in $\mathbb{R}^3$, intersecting each other at a line. Here we will try to use Method 3:

- $U$ is 2-dimensional (3 free variables - 1 constraint)
- $V$ is 2-dimensional (3 free variables - 1 constraint)
- $U, V \subseteq \mathbb{R}^3$, which is 3-dimensional.

Because

$$\dim U + \dim V = 4 > \dim \mathbb{R}^3,$$

the sum $U + V$ is not a direct sum.

* There are several ways to check if $V_1 + V_2 + \ldots + V_n$ is a direct sum. Below are some (usually) quick ways:

1) Find a basis $B_1$ of $V_1$, ..., $B_n$ of $V_n$. Then check if (or show that) the concatenation $B_1 \cup B_2 \cup \ldots \cup B_n$ is linearly independent.

2) Let $v_1 + v_2 + \ldots + v_n = 0$ for some $v_1 \in V_1, \ldots, v_n \in V_n$. Check if $v_1 = v_2 = \ldots = v_n = 0$.

3) If $V_1, \ldots, V_n \subseteq W$ and $\dim V_1 + \ldots + \dim V_n > \dim W$ then $V_1 + \ldots + V_n$ is not a direct sum.

Warning: for $n > 2$, the fact that $V_1 + \ldots + V_n$ is a direct sum is not equivalent to the fact that the intersection $V_1 \cap V_2 \cap \ldots \cap V_n$ is equal to $\{0\}$. 

Ex:

We see that \( V_1 \cap V_2 = V_1 \cap V_3 = V_1 \cap V_2 \cap V_3 = \{0\} \).
However, \( V_1 + V_2 + V_3 \) is not a direct sum because \( (V_1 + V_2) \cap V_3 \neq \{0\} \).
This intersection is a line.

Recall that for \( V_1 + \ldots + V_n \) to be a direct sum, one must have \( V_i \cap (V_1 + \ldots + V_{i-1} + V_{i+1} + \ldots + V_n) = \{0\} \) for all \( i = 1, 2, \ldots, n \).
Ex:

\[
V_1 = \{ (x,y,z) : x+y+z = 0 \} \\
V_2 = \{ (x,y,z) : x+2y = y+z = 0 \} \\
V_3 = \{ (x,y,z) : x = 2y = 3z \}. 
\]

Show that \( V_1 \oplus V_2 \oplus V_3 = \mathbb{R}^3 \).

The problem asks us to show 2 things:

1. \( V_1 + V_2 + V_3 \) is a direct sum.
2. \( V_1 + V_2 + V_3 = \mathbb{R}^3 \)

Let us show (1):

We will find a basis for each \( V_1, V_2, V_3 \):

\( V_1 = \{ (x,-x,0) : x \in \mathbb{R} \} = \text{span} \{ (1,-1,0) \} \)
has basis \( B_1 = \{ (1,-1,0) \} \).

\( V_2 = \{ (x,\frac{x}{2}, \frac{x}{2}) : x \in \mathbb{R} \} = \text{span} \{ (1,-1,1) \} \)
has basis \( B_2 = \{ (1,-1,1) \} \).

\( V_3 = \{ (x, \frac{x}{2}, \frac{x}{2}) : x \in \mathbb{R} \} = \text{span} \{ (1,1,1) \} \)
has basis \( B_3 = \{ (1,1,1) \} \).

Concatenation of bases:

\( B_1 \cup B_2 \cup B_3 = \{ (1,-1,0), (1,-\frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2}) \} \).
These vectors are linearly independent because

\[
\begin{vmatrix}
1 & 1 & 0 \\
1 & -\sqrt{2} & \sqrt{2} \\
1 & \sqrt{2} & \sqrt{2}
\end{vmatrix} \neq 0.
\]

Therefore, \( V_1 + V_2 + V_3 \) is a direct sum.

Show (2):

We know that \( V_1 + V_2 + V_3 \) is a subspace of \( \mathbb{R}^3 \).
We also know from above that \( \dim(V_1 + V_2 + V_3) = 3 \).
Therefore, \( V_1 + V_2 + V_3 = \mathbb{R}^3 \).

Ex:

Let \( A \in M_{n \times n}(\mathbb{R}) \). Consider the following sets:

\[
V_1 = \{ v \in \mathbb{R}^n : Av = v_3 \}
\]

\[
V_2 = \{ v \in \mathbb{R}^n : Av = 2v_3 \}
\]

\[
V_3 = \{ v \in \mathbb{R}^n : Av = 3v_3 \}.
\]

Show that \( V_1 + V_2 + V_3 \) is a direct sum.

In this problem, it is hard to use the first method (i.e. find a basis for each \( V_1, V_2, V_3 \)) because \( A \) is quite a general matrix. We will try the second method:

Let \( v_1 \in V_1, v_2 \in V_2, v_3 \in V_3 \) be such that \( v_1 + v_2 + v_3 = 0 \).
We will show that \( v_1 = v_2 = v_3 = 0 \).

Multiplying both sides of the equation by \( A \):

\[ A(v_1 + v_2 + v_3) = A0 = 0 \]

which is equivalent to

\[ Av_1 + Av_2 + Av_3 = 0 \]

which is equivalent to

\[ v_1 + 2v_2 + 3v_3 = 0. \]
Multiplying this equation by $A$, we get
$$A(v_1 + 2v_2 + 3v_3) = A0 = 0$$
which is equivalent to
$$Av_1 + 2Av_2 + 3Av_3 = 0$$
which is equivalent to
$$v_1 + 4v_2 + 9v_3 = 0.$$  
We have got three equations:

$$\begin{align*}
v_1 + v_2 + v_3 &= 0 \\
v_1 + 2v_2 + 3v_3 &= 0 \\
v_1 + 4v_2 + 9v_3 &= 0
\end{align*}$$

Note that $v_1, v_2, v_3$ are vectors, not numbers. However, one can use familiar methods to solve for $v_1, v_2, v_3$, for example by substitution and elimination. If $v_1, v_2, v_3$ were numbers, we can write the system in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now that $v_1, v_2, v_3$ are vectors, we only need to adjust this form slightly to make it correct:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiply both sides by the inverse of the constant matrix:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Therefore, $v_1 = v_2 = v_3 = 0$. We conclude that $v_1 + v_2 + v_3$ is a direct sum.