The statements \( V_1 \oplus V_2 \oplus \ldots \oplus V_n = V \) means two things:
1. The sum \( V_1 + V_2 + \ldots + V_n \) is a direct sum,
2. \( V_1 + V_2 + \ldots + V_n = V \).

**Ex:**

Let \( V \) be a vector space over a field of numbers \( F \), which could be \( \mathbb{Q} \), \( \mathbb{R} \), or \( \mathbb{C} \). For each \( v \in V \), we use the notation
\[
F_v := \{ cv : c \in F \} = \text{span}\{v\}.
\]

If \( v \neq 0 \) then \( F \) is a 1-dimensional subspace with basis \( \{v\} \).

Suppose \( B = \{v_1, v_2, \ldots, v_n\} \) is a basis of \( V \). Then
\[
V = F_{v_1} \oplus F_{v_2} \oplus \ldots \oplus F_{v_n}. \tag{\*}
\]
In other words, \( V \) is "decomposed" into \( n \) one-dimensional subspaces. Why is \( (\*) \) true?

We need to show two things:

1. The sum \( F_{v_1} + \ldots + F_{v_n} \) is a direct sum.
2. \( F_{v_1} + \ldots + F_{v_n} = V \).

To show (1), we take a basis of each subspace \( F_{v_i} \).

Choose \( B_i = \{v_i\} \). The concatenation is
\[
B_1 \cup B_2 \cup \ldots \cup B_n = \{v_1, v_2, \ldots, v_n\}.
\]
This set is linearly independent because it is a basis of \( V \).

Therefore, (1) is true.

To show (2), we notice that \( F_{v_1} + \ldots + F_{v_n} \) is a subspace of \( V \). Because \( F_{v_1} + \ldots + F_{v_n} \) is a direct sum,
\[
\dim (F_{v_1} + \ldots + F_{v_n}) = \dim F_{v_1} + \ldots + \dim F_{v_n}\]

= n = \dim V.

Therefore, \( E V_1 + \ldots + E V_n = V \).

Ex.

Consider the following subspaces of \( M_{2 \times 2}(\mathbb{R}) \):

\[ V_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = b + c = 0 \right\} \]

\[ V_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + c = b = d = 0 \right\} \]

\[ V_3 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = c = d = 0 \right\} \]

Show that \( V_1 \oplus V_2 \oplus V_3 = M_{2 \times 2}(\mathbb{R}) \).

Our strategy is to convert this problem into a problem in \( \mathbb{R}^4 \) by using coordinates. Consider the standard basis of \( M_{2 \times 2}(\mathbb{R}) \):

\[ B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \]

Each vector \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) of \( M_{2 \times 2}(\mathbb{R}) \) corresponds to a vector \( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \) of \( \mathbb{R}^4 \).

\[ V_1 \text{ has a basis } B_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}. \]

\[ V_2 \text{ has a basis } B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}. \]
\( V_1 \) has a basis \( B_3 = \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \} \)

We convert the problem on \( M_{22}(\mathbb{R}) \) to a problem on \( \mathbb{R}^4 \) as follows.

\( V_1 \) corresponds to a subspace \( V_1' \) of \( \mathbb{R}^4 \) with basis
\[
B_1' = \{ \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \}
\]

\( V_2 \) corresponds to a subspace \( V_2' \) of \( \mathbb{R}^4 \) with basis
\[
B_2' = \{ \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \}
\]

\( V_3 \) corresponds to a subspace \( V_3' \) of \( \mathbb{R}^4 \) with basis
\[
B_3' = \{ \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \}
\]

We want to show \( V_1' \oplus V_2' \oplus V_3' = \mathbb{R}^4 \).

This means that we need to show two things:

1. \( V_1' + V_2' + V_3' \) is a direct sum,
2. \( V_1' + V_2' + V_3' = \mathbb{R}^4 \).

Show (1):

Concatenate the bases:
\[
\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\]

\( B_1' \quad B_2' \quad B_3' \)

\[
= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Thus, \( B_1' \cup B_2' \cup B_3' \) is linearly independent.

Show (2):

We know that \( V_1' + V_2' + V_3' \) is a subspace of \( \mathbb{R}^4 \) and
\[ \dim (V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 \quad \text{(due to direct sum)} \]
\[ = 2 + 1 + 1 \]
\[ = 4 \]
\[ = \dim \mathbb{R}^4. \]
Therefore, \( V_1 + V_2 + V_3 = \mathbb{R}^4 \).

* Invariant subspaces:
Recall the definition: let \( f: V \to V \) be a linear map. Note that \( f \) goes from \( V \) to itself. A subspace \( W \subset V \) is called invariant under \( f \) if \( f(W) \subset W \).

Note:
\[ f(W) \triangleq \{ f(x) : x \in W \} \]
(the set of the images of vectors in \( W \) under \( f \)).

Intuitively, if \( W \) is invariant under \( f \) then \( f \) can be "localized" to \( W \).

To show \( f(W) \subset W \), one starts by writing:
"Take \( x \in W \). We want to show \( f(x) \in W \)."

See an example on the worksheet.