Let \( f : V \rightarrow V \) be a linear map. How to check if \( f \) is diagonalizable?

Suppose that \( f \) is diagonalizable. Then \( V \) can be decomposed into \( 1 \)-dim invariant subspaces under \( f \):

\[
V = V_1 \oplus V_2 \oplus \ldots \oplus V_n
\]

Each \( V_k \) is the span of \( \{v_k\} \) for some \( v_k \neq 0 \).

\( V_k \) corresponds to an eigenvalue \( \lambda_k \). \( f(v_k) = \lambda_k v_k \). If we put

\[
E(\lambda_k) = \{ v \in V : f(v) = \lambda_k v \}
\]

then \( V_k \) is contained in \( E(\lambda_k) \). Note that \( E(\lambda_k) \) is called the eigenspace corresponding to eigenvalue \( \lambda_k \).

Let \( \lambda_1, \lambda_2, \ldots, \lambda_m \) be distinct eigenvalues of \( f \). We have reasoned that each of \( V_1, V_2, \ldots, V_n \) is contained in one of \( E(\lambda_1), E(\lambda_2), \ldots, E(\lambda_m) \). Thus,

\[
V_1 + V_2 + \ldots + V_n \subseteq E(\lambda_1) + \ldots + E(\lambda_m) = V
\]

Because the RHS is a subspace of \( V \), we get

\[
V = E(\lambda_1) + \ldots + E(\lambda_m).
\]

Note that the RHS is also a direct product (without any assumption on \( f \) except that \( f \) is linear). We proved a special case of this in Lecture 20 (11/15/2019).

In conclusion, if \( f \) is diagonalizable then \( V = E(\lambda_1) \oplus \ldots \oplus E(\lambda_m) \).
Now suppose that \( V = E(\lambda_1) \oplus E(\lambda_2) \oplus \ldots \oplus E(\lambda_m) \). We show that \( f \) is diagonalizable.

Let \( B_k = \{ v_{i_1}^{(k)}, v_{i_2}^{(k)}, \ldots, v_{i_r}^{(k)} \} \) be a basis of \( E(\lambda_k) \). Then \( E(\lambda_k) \) is 1-dim and invariant under \( f \).

Then \( V = E(\lambda_1) \oplus E(\lambda_2) \oplus \ldots \oplus E(\lambda_m) \)

\[
= (Fv_1^{(1)} \oplus \ldots \oplus Fv_r^{(1)}) \oplus (Fv_1^{(2)} \oplus \ldots \oplus Fv_r^{(2)}) \oplus \ldots \oplus (Fv_1^{(m)} \oplus \ldots \oplus Fv_r^{(m)})
\]

This shows that \( V \) is decomposed into 1-dim invariant subspaces.

In conclusion, we have showed:

**Theorem:**
Let \( f: V \to V \) be a linear map, where \( V \) is a vector space over \( F \). Let \( \lambda_1, \ldots, \lambda_m \) be all distinct eigenvalues of \( f \). Then \( f \) is diagonalizable if and only if \( V = E(\lambda_1) \oplus E(\lambda_2) \oplus \ldots \oplus E(\lambda_m) \).

In problem solving, we only need to check if \( \dim V = \dim E(\lambda_1) + \ldots + \dim E(\lambda_m) \).

**Procedure to check if \( f: V \to V \) is diagonalizable:**

1) Find all the distinct eigenvalues of \( f \), called \( \lambda_1, \ldots, \lambda_m \).
2) Find a basis \( B_k \) for eigenspace \( E(\lambda_k) \).
3) Check if \( \dim V = \dim \text{E}(d_1) + \ldots + \dim \text{E}(d_m) \)

If they are not equal, conclude that \( f \) is not diagonalizable.
If they are equal then \( f \) is diagonalizable. The basis

\[ B = B_1 \cup B_2 \cup \ldots \cup B_m \]

diagonalizes \( f \) because \( [f]_B \) is a diagonal matrix.

\[
[f]_B = \begin{bmatrix}
[f(x_1^{(1)})]_B & [f(x_2^{(1)})]_B & \cdots & [f(x_i^{(1)})]_B & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\alpha_1 & 0 & \cdots & 0 \\
0 & \alpha_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_m \\
\end{bmatrix}
\]