1. Let $U$, $V$, and $W$ be subspaces of $\mathbb{R}^4$ defined by

$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2, \ x_3 = x_4\}$

$V = \text{span}(\{(1, 0, 0, 1), (0, 1, 1, 0)\})$

$W = \{(0, x, 0, y) : x, y \in \mathbb{R}\}$

(a) Is $U + V$ a direct sum of $U$ and $V$?

**Solution:** No. Notice that $(1, 1, 1, 1) \in U$

and $(1, 1, 1, 1) \in V$ (since $(1, 1, 1, 1) = (1, 0, 0, 1) + (0, 1, 1, 0)$).

Therefore $(1, 1, 1, 1) \in U \cap V$, so $U \cap V \neq \{(0, 0, 0, 0)\}$.

(b) Is $V + W$ a direct sum of $V$ and $W$?

**Solution:** Yes. Let $(0, x, 0, y) \in W$. If $(0, x, 0, y)$ is also in $U$, then

$0 = x_1 = x_2 = x$ and $0 = x_3 = x_4 = y$.

Therefore $V \cap W = \{(0, 0, 0, 0)\}$.

2. Let

$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2, \ x_4 = x_1 + x_3\}$

Find a subspace $W$ of $\mathbb{R}^4$ such that $V \oplus W = \mathbb{R}^4$.

**Solution:** First find a basis for $V$:

$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2, \ x_4 = x_1 + x_3\}$

$= \{(x_1, x_1, x_3, x_4) \in \mathbb{R}^4 : x_4 = x_1 + x_3\}$

$= \{(x_1, x_1, x_3, 1) \in \mathbb{R}^4\}$

$= \{x_1(1,1,0,1) + x_3(0,0,1,1) : x_2, x_3 \in \mathbb{R}\}$

so $\{(1,1,0,1),(0,0,1,1)\}$ is a basis for $V$. Now form the matrix $A$ using the basis vectors as rows:

$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Notice that $A$ is already in reduced row echelon form. Find the non-pivot columns of RREF($A$):

$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\uparrow \uparrow$

These are columns 2 and 4. Let $W$ be the span of standard basis vectors with a 1 in the coordinates corresponding to the non-pivot columns:

$W = \text{span}(\{e_2, e_4\}) = \text{span}(\{(0,1,0,0),(0,0,0,1)\}) = \{(0,x,0,y) : x, y \in \mathbb{R}\}$

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We now need to check that \( V \cap W = \{(0, 0, 0, 0)\} \). Let \((x_1, x_2, x_3, x_4) \in V \cap W \). Then

\[ x_1 = x_3 = 0 \quad \text{since} \quad (x_1, x_2, x_3, x_4) \in W, \]
\[ x_2 = x_1 = 0 \quad \text{since} \quad (x_1, x_2, x_3, x_4) \in V, \]
\[ x_4 = x_1 + x_3 = 0 \quad \text{since} \quad (x_1, x_2, x_3, x_4) \in V. \]

Therefore \( V \cap W = \{(0, 0, 0, 0)\} \), so \( U + W = U \oplus W \) is a direct sum of \( U \) and \( W \). To see that \( U \oplus W = \mathbb{R}^4 \), notice that

\[ \dim(U \oplus W) = \dim(U) + \dim(W) = 2 + 2 = 4 = \dim(\mathbb{R}^4) \]

Since \( U \oplus W \) is a subspace of \( \mathbb{R}^4 \) of equal dimension, \( U \oplus W = \mathbb{R}^4 \).

3. Find a vector space \( V \) with subspaces \( U_1, U_2, \) and \( W \) such that \( U_1 \oplus W = U_2 \oplus W \) but \( U_1 \neq U_2 \).

**Solution:** Let

\[ V = \mathbb{R}^2 \]
\[ U_1 = \{(x, 0) : x \in \mathbb{R}\} \]
\[ U_2 = \{(0, y) : y \in \mathbb{R}\} \]
\[ W = \{(z, z) : z \in \mathbb{R}\}. \]

It is not hard to check that

\[ U_1 + W = \text{span}\{(1, 0), (1, 1)\} = \mathbb{R}^2 \]

and

\[ U_2 + W = \text{span}\{(0, 1), (1, 1)\} = \mathbb{R}^2 \]

so \( U_1 + W = U_2 + W \).

Now notice that if \((a, b) \in U_1 \cap W \) then \( b = 0 \) (since \((a, b) \in U_1\)) and \( a = b = 0 \) (since \((a, b) \in W\)). Therefore \( U_1 + W \) is a direct sum of \( U_1 + W \). A similar argument shows that \( U_1 + W \) is a direct sum of \( U_1 \) and \( W \).

Finally, notice that \( U_1 \neq U_2 \), since for example \((1, 0) \in U_1 \) but \((1, 0) \notin U_2 \).

4. Consider \( \mathbb{C} \) as a vector space over the field \( F = \mathbb{C} \). Prove that the map \( f : \mathbb{C} \to \mathbb{C} \) given by \( f(z) = \overline{z} \) is not linear.

**Solution:** It can be checked that \( f \) is additive. Therefore we want to find \( z \in \mathbb{C} \) and \( \lambda \in F = \mathbb{C} \) such that \( f(\lambda z) \neq \lambda f(z) \). Let

\[ z = 1 \quad \text{and} \quad \lambda = i \]

Then

\[ f(\lambda z) = f(i) = -i \]

and

\[ \lambda f(z) = i \cdot f(1) = i \cdot 1 = i, \]

so \( f(\lambda z) \neq \lambda f(z) \).