1. Consider the following subspaces of $\mathbb{R}^4$:

$$V_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = 0\},$$

$$V_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3 = 0\},$$

$$V_3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_2 = x_3 = x_4 = 0\},$$

$$V_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3 = x_4 = 0\}.$$

Which of the following sums are direct sums: (a) $V_1 + V_2$; (b) $V_2 + V_3 + V_4$; (c) $V_1 + V_3 + V_4$?

A basis of $V_1$ is $B_1 = \{(0,1,0,0), (0,0,0,1)\}$. 

A basis of $V_2$ is $B_2 = \{(0,1,0,0), (0,0,0,1)\}$. 

A basis of $V_3$ is $B_3 = \{(0,1,0,0)\}$. 

A basis of $V_4$ is $B_4 = \{(0,1,0,0)\}$. 

$V_1 + V_2$ is not a direct sum because $(0,1,0,0) \in V_1 \cap V_2$.

To see if $V_2 + V_3 + V_4$ is a direct sum, we concatenate $B_2, B_3, B_4$:

$$\begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

These 4 vectors are not linearly independent. Thus, $V_2 + V_3 + V_4$ is not a direct sum.

By a similar method, we can see that $V_1 + V_3 + V_4$ is a direct sum.
2. Let \( F : P_3 \rightarrow P_3 \) be a linear map given by \( F(u) = xu' \). Consider the following subspaces \( U = \text{span}\{x, 1\} \) and \( V = \text{span}\{x^2 - 1, x + 1\} \). Check whether \( U \) and \( V \) are invariant under \( F \).

* Check if \( U \) is invariant under \( F \):

Let \( u \in U \). We want to check if \( F(u) \in U \).

By the def. of \( U \), we can write \( u = ax + b \) for some \( a, b \in \mathbb{R} \).

Then \( F(u) = xu' = ax \).

Then \( F(u) \in U \). Thus, \( U \) is invariant under \( f \).

* Check if \( V \) is invariant under \( F \):

Let \( v \in V \). We want to check if \( F(v) \in V \).

By the definition of \( V \), we can write \( v = a(x^2 - 1) + b(x + 1) = ax^2 + bx - a + b \).

Then \( F(v) = xv' = x(2ax + b) = 2ax^2 + bx \).

For \( a = 0 \) and \( b = 1 \), we have \( v = x + 1 \) and \( F(v) = x \).

We will show that \( x \notin V = \text{span}\{x^2 - 1, x + 1\} \).

Suppose by contradiction that \( x \in V \). Then there are \( c, d \in \mathbb{R} \) such that

\[
x = c(x^2 - 1) + d(x + 1).
\]

Equivalently,

\[
cx^2 + (d - 1)x - c + d = 0 \quad \forall x \in \mathbb{R}
\]

This only happens if \( \begin{cases} c = 0 \\ d - 1 = 0 \\ c + d = 0 \end{cases} \) however, this system is inconsistent.

Therefore, \( F(v) = x \notin V \). We conclude that \( V \) is not invariant under \( F \).