Homework 7
Due 11/25/2019

1. Given a function $f$ on some interval, say $[-1, 1]$, and an integer $N > 1$, we are interested in the question: what set of sample points $\{x_1, x_2, \ldots, x_n\}$ on $[-1, 1]$ should we choose so that the interpolation polynomial $P_n$ can best approximate function $f$? Note that the number of sample points $n$ is fixed. We are free to choose the sample points.

To investigate this question, let us consider an example $f(x) = \frac{1}{1 + 10x^2}$ and $n = 11$. Consider two different ways of sampling:

- Evenly spaced $-1 = x_1 < x_2 < \ldots < x_n = 1$,
- Unevenly spaced $z_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ for $k = 1, 2, \ldots, n$.

(a) Use the command Plot to sketch each set of sample points on the interval $[-1, 1]$.
(b) Let $P_n$ be the interpolation polynomial corresponding to the set of data points $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$. Plot $P_n$ and $f$ on the same graph.
(c) Let $Q_n$ be the interpolation polynomial corresponding to the set of data points $(z_1, f(z_1)), \ldots, (z_n, f(z_n))$. Plot $Q_n$ and $f$ on the same graph.
(d) Based on the graphs, is one way of sampling significantly better than the other? Give a rough explanation for your observation.
(e) The same questions in Parts (b), (c), (d) but for $f(x) = \cos x$.

2. Interpolation gives an alternative method to approximate a function $f$ by polynomials (other than Taylor approximation method). In this exercise, we investigate error estimates of this method. Let $f(x) = e^x \sin\left(\frac{x}{2}\right)$.

For evenly spaced sample points $0 = x_1 < x_2 < \ldots < x_n = 4$, let $P_n$ be the corresponding interpolation polynomial.

(a) Show that $|f'(x)| \leq e^{x/2}$ and $|f''(x)| \leq e^{x/2}$.
(b) It is known that (you don’t have to verify) $|f^{(k)}(x)| \leq e^{x/2}$ for any $x \in \mathbb{R}$ and $k \geq 1$. Find $n$ such that $|f(x) - P_n(x)| \leq 10^{-4}$ for all $x \in [0, 4]$.
(c) Find $n$ such that the integral $\int_0^4 P_n(x) dx$ approximates $\int_0^4 f(x) dx$ with error not exceeding $10^{-3}$.

Hint: use the inequality

$$\left| \int_a^b (f(x) - g(x)) dx \right| \leq \int_a^b |f(x) - g(x)| dx.$$