Denote \( I = \int_0^2 \frac{1}{4 + x^2} \, dx \).

**Problem 1.**

Find the exact value of \( I \).

**Solution**

\[
\int_0^2 \frac{1}{4 + x^2} \, dx = \left. \frac{1}{2} \arctan \left( \frac{x}{2} \right) \right|_0^2 = \frac{1}{2} \arctan(1) - \arctan(0) = \frac{\pi}{8}
\]

**Problem 2.**

For a generic positive integer \( n \) we take \( n + 1 \) equally spaced sample points indexed by \( x_0, x_1, \ldots, x_n \) on the interval \([0, 2]\). Denote by \( L_n, R_n, M_n, T_n \) the Riemann sums corresponding to the left-point, right-point, midpoint, and trapezoid rule. Use sigma notation to write a formula for each \( L_n, R_n, M_n, T_n \).

**Solution**

\[
L_n = \sum_{i=0}^{n-1} \frac{2}{n} f(x_i) = \sum_{i=0}^{n-1} \frac{2}{n} \frac{1}{4 + x_i^2} = \sum_{i=0}^{n-1} \frac{2}{n} \frac{1}{4 + \left( \frac{i}{n} \right)^2} = \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{n + \frac{i^2}{n}}
\]

\[
R_n = \sum_{i=0}^{n} \frac{2}{n} f(x_i) = \frac{1}{2} \sum_{i=0}^{n} \frac{1}{n + \frac{i^2}{n}}
\]

Note that the indexing has changed between \( L_n \) and \( R_n \).

\[
M_n = \sum_{i=0}^{n-1} \frac{2}{n} \frac{1}{4 + \left( \frac{2i+1}{n} \right)^2} = \sum_{i=0}^{n-1} \frac{2}{4n + \frac{(2i+1)^2}{n^2}} = \frac{n}{\sum_{i=0}^{n-1} 4n^2 + (2i+1)^2}
\]

\[
T_n = \sum_{i=0}^{n-1} \frac{2}{n} f(x_i) + f(x_{i+1}) = \sum_{i=0}^{n-1} \frac{1}{n} \left( \frac{1}{4 + \left( \frac{i}{n} \right)^2} + \frac{1}{4 + \left( \frac{i+1}{n} \right)^2} \right) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{n^2 (2i(i+1) + 2n^2 + 1)}{4 (i^2 + n^2) ((i+1)^2 + n^2)}
\]

(Please simplify your answer so the sum is entirely in terms of an index variable (\( i \) above) and \( n \).)

**Problem 3.**

Which of these three methods gives the best approximation of \( I \) when \( n = 4 \)?

**Solution**

We can use the above formulas to compute the approximations, then find the error bounds.

\[
|L_4 - \pi/8| \approx 0.029947877124805
\]

\[
|R_4 - \pi/8| \approx 0.032552022875195
\]

\[
|M_4 - \pi/8| \approx 6.509831005186983 \times 10^{-4}
\]

\[
|T_4 - \pi/8| \approx 0.001302022875195
\]

\( M_4 \) gives the best approximation of the four choices.
Problem 4.

Write matlab code to compute $L_n$, $R_n$, $M_n$, and $T_n$ for $n = 8, 16, 32, 64$.

Solution

We do this in Matlab. For simplicity, we write one script for each method which allows us to easily change $n$ (line 3).

Left point method:

```matlab
a = 0;
b = 2;
n = 4;
xvals = linspace(a,b,n+1); % Generate n+1 points
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii))*(b-a)/n;
end
disp(total)
error = total - pi/8;
disp(error)
function out = objective(in)
    out = 1./(4 + in.^2);
end
```

Right point method:

```matlab
a = 0;
b = 2;
n = 4;
xvals = linspace(a,b,n+1); % Generate n+1 points
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii+1))*(b-a)/n;
end
disp(total)
error = total - pi/8;
disp(error)
function out = objective(in)
    out = 1./(4 + in.^2);
end
```

Trapezoidal method:

```matlab
a = 0;
b = 2;
n = 4;
xvals = linspace(a,b,n+1); % Generate n+1 points
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii) + yvals(ii+1))/2*(b-a)/n;
end
disp(total)
```
```matlab
error = total - pi/8;
disp( error)
function out = objective(in)
    out = 1./(4 + in.^2);
end
```

Midpoint method:
```matlab
a = 0;
b = 2;
n = 4;
xvals = linspace(a,b,n+1); % Generate n+1 points
xvals = xvals + 1/n; % Shift xvalues by one half
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii))*(b-a)/n;
end
disp( total)
error = total - pi/8;
disp( error)
function out = objective(in)
    out = 1./(4 + in.^2);
end
```

Problem 5.

Find values of $n$ such that the error for each left point, right point, midpoint, and trapezoidal rule approximations are bounded by $\epsilon = 0.0001$.

Solution

We need to find bounds on $K$ and $\widetilde{K}$.

$$f'(x) = \frac{-2x}{(4+x^2)^2} \implies |f'(x)| = \frac{2x}{(4+x^2)^2}, \quad x \in [0, 2]$$

$$f''(x) = \frac{8x - 6x^3}{(x)(4+x^2)^3}, \quad f''(x) = 0 \implies 8x = 6x^3 \implies x = \pm \frac{2}{\sqrt{3}}$$

The point $x = \frac{2}{\sqrt{3}}$ is inside the desired interval, and maximizes the absolute value of $f'(x)$ (as $f'(x) > f'(0)$ for $x > 0$). Then we can bound $|f'(x)|$ on the interval by $K = \frac{3\sqrt{3}}{64} \approx 0.0811898816$, or choose any value larger. Likewise,

$$f'''(x) = \frac{24x(x^2 - 4)}{(4+x^2)^4}, \quad f'''(x) = 0 \implies x^2 - 4 = 0 \implies x \in \{-2, 0, 2\}$$

And we check the endpoints to find that $f'''(0) > f'''(2)$ ($-2$ is not in the interval). So $|f''(x)| \leq \frac{1}{8} = \widetilde{K}$. You can choose any bound greater than this that you can justify. One simple method is as follows:

$$|f'(x)| = \frac{2x}{(4+x^2)^2} \leq \frac{2(2)}{(4+0^2)^2} = \frac{1}{4}.$$ 

$$|f''(x)| = \frac{|8 - 6x^2|}{(4+x^2)^3} \leq \frac{8 + 6x^2}{(4+x^2)^3} \leq \frac{8 + 6(2)^2}{(4+0^2)^3} = \frac{1}{2}.$$
Left point and right point methods have the same error bound.

\[ e_n^{(L)}, e_n^{(R)} \leq \frac{K(2)^2}{2n} \leq \frac{(1/4)(2)^2}{2n} = \frac{1}{2n} \leq 0.0001 \Rightarrow n \geq 5000. \]

\[ e_n^{(M)} \leq \frac{\tilde{K}(2)^3}{24n^2} \leq \frac{(1/2)(2)^3}{24n^2} = \frac{1}{6n^2} \leq 0.0001 \Rightarrow n \geq 41. \]

So choosing \( n \geq 41 \) is sufficient. Lastly,

\[ e_n^{(T)} \leq \frac{\tilde{K}(2)}{12n^2} = \frac{(1/2)(2)^3}{12n^2} = \frac{1}{3n^2} \leq 0.0001 \Rightarrow n \geq 58. \]

So choosing \( n \geq 58 \) is sufficient.