Last time, we posed the problem of evaluating the integral \( \int_0^1 \cos(x^3) \, dx \) numerically.

This integral can be approximated by Riemann sum.

\[
\int_0^1 f(x) \, dx \approx \sum_{k=0}^{n-1} \text{area of rectangle } k\text{'th}
\]

\[
= \sum_{k=0}^{n-1} \frac{1}{n} f\left( \frac{k}{n} \right)
\]

\[
= \frac{1}{n} \sum_{k=0}^{n-1} \cos\left( \frac{k^2}{n^2} \right)
\]

**Comment:**

The idea of approximating the area of a general shape by the area of simpler shapes dates back to Archimedes (if not earlier). He used this idea to approximate the area of a circle.

Note that only have in hand the area of a square. Then area of rectangle is obtained by concatenating many squares together.

right triangle  \hspace{1cm} \text{triangle}  \hspace{1cm} \text{quadrilateral}  \hspace{1cm} \text{circle (approximation)}
With $n = 1000$, 
\[
\int_0^1 \cos(x^2)\,dx \approx \frac{1}{1000} \sum_{k=0}^{999} \cos\left(\frac{k^2}{1000}\right).
\] (*)

Before the age of computer, one can only rely on desk calculators for basic operations such as addition, subtraction, multiplication and division. Thus, a desk calculator can only evaluate polynomials and rational functions (quotient of two polynomials). The approximation (*) would not be satisfying at that time because it involves transcendental function ($\cos$).

An approximation method we already knew is approximation by Taylor polynomials. Recall:

If $f$ is a $(n+1)$st differentiable on an interval containing $x_0$, then
\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)
\]
where
\[
R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}
\]
c is some number between $x_0$ and $x$.

This is known as Lagrange theorem. Taylor appr. helps us evaluate approximately quite general functions:
\[
f(x) = p_n(x) + R_n(x)
\]
computable error, hopefully estimable.

If one knows how to control the size of $f^{(n+1)}$ then the error term $R_n(x)$ is under control.
$\sqrt{8}$ with precision $10^{-9}$.

We know that $\sqrt{9} = 3$. Let's consider function

$$f(x) = \sqrt{x}$$

Then

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f''(x) = \frac{1}{2} \left( -\frac{1}{2} \right) x^{-3/2}$$

$$f'''(x) = \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) x^{-5/2}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \cdots \left( -\frac{2n-3}{2} \right) x^{-\frac{2n-1}{2}}$$

$$n$$ terms

$$= \frac{1}{2} \left( \frac{1}{2}-1 \right) \left( \frac{1}{2}-2 \right) \cdots \left( \frac{1}{2}-(n-1) \right) x^{-\frac{1}{2}-(n-1)}$$

Therefore,

$$f^{(n)}(g) = \frac{1}{2} \left( \frac{1}{2}-1 \right) \left( \frac{1}{2}-2 \right) \cdots \left( \frac{1}{2}-(n-1) \right) g^{-\frac{1}{2}-(n-1)}$$

Now we can apply Taylor approx. to $f$ about $x_0 = g$, with $x_8$

$$f(x) = f(g) + \sum_{k=1}^{n} \frac{f^{(k)}(g)}{k!} (x-g)^k + R_n(x)$$

Let us simplify $p_n(8)$.

$$p_n(8) = 3 + \sum_{k=1}^{n} \frac{f^{(k)}(g)}{k!} (-1)^k$$

This sum is programmable in Matlab.

Let us estimate the error by Lagrange's theorem:

$$R_n(8) = \frac{f^{(n+1)}(c)}{(n+1)!} (8-g)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}$$
Here \( c \) is some number between 8 and 9.

\[
\frac{f^{(n+1)}(c)}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right) \ldots \left( \frac{1}{2} - n \right) c^{-\frac{1}{2} - n}
\]

The size of \( f^{(n+1)}(c) \) is bounded by

\[
\left| f^{(n+1)}(c) \right| \leq \frac{1}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right) \ldots \left( \frac{1}{2} - n \right) 8^{-\frac{1}{2} - n}
\]

\[
\leq 1 \leq 2 \leq n
\]

\[
= \frac{1}{2} n! 8^{-\frac{1}{2} - n}
\]

\[
< \frac{1}{2} n! 8^{-n}
\]

Then

\[
\left| R_n(8) \right| = \left| f^{(n+1)}(c) \right| \leq \frac{1}{2} n! 8^{-n} \leq \frac{1}{2(n+1)!} 8^{-n} = \frac{1}{2(n+1)!} 8^{-n}
\]

This number goes to 0 quite rapidly as \( n \) goes to infinity. If we have \( \frac{1}{2(n+1)!} 8^{-n} < 10^{-9} \) then \( \left| R_n(8) \right| \) is guaranteed to be under \( 10^{-9} \). We can check with a calculator that \( n = 10 \) will do it.