Last time we derived some error estimates for the left point, right point and midpoint rule. Accordingly, if we put
\[
\epsilon_n = \left| \int_a^b f(x) \, dx - \text{Riemann sum} \right|
\]

Then
\[
\epsilon_n \leq \frac{M(b-a)^2}{2n} \quad \text{where} \quad M = \max_{[a,b]} |f(x)|.
\]

This suggests that as \( n \to \infty \), \( \epsilon_n \) goes to 0 at a rate \( \frac{1}{n} \) (or faster).

This is a polynomial rate, as opposed to exponential rate \( (2^{-n}, e^{-n}, \ldots) \).

It is natural to ask what is the order of convergence of \( \epsilon_n \) to 0?

It is easier for us to find the order of convergence of \( a_n \) rather than \( \epsilon_n \).

We want to find numbers \( p \) and \( C \) such that \( a_{n+1} \leq Ca_n^p \).

Because
\[
a_{n+1} \sim \frac{1}{n+1}
\]
\[
a_n^p \sim \frac{1}{n^p}
\]

we see that for \( a_{n+1} \leq Ca_n^p \) for \( n \) large, \( p \) is at most 1.

Therefore, \( a_n \) converges to 0 at order \( p=1 \).

For the left/right point rule, \( \frac{1}{n} \) is the optimal rate of decay of \( \epsilon_n \).

In other words, one can find an example of a function \( f \) such that \( \epsilon_n \) goes to 0 no faster than \( \frac{1}{n} \).
* For midpoint rule, however, \( \frac{1}{n} \) is not an optimal rate of decay of \( e_n \). By a more subtle error estimate (using Taylor expansion of degree 1), one can show that

\[
e_n \leq \frac{\tilde{M} (b-a)^3}{24} \frac{1}{n^2} \quad \text{where} \quad \tilde{M} = \max_{[a,b]} |f''(x)|
\]

This shows that \( e_n \) in fact goes to 0 at a rate of \( \frac{1}{n^2} \) (or faster). However, the order of convergence of \( b_n \) is still equal to 1.

* For trapezoidal rule,

\[
e_n \leq \frac{\tilde{M} (b-a)^3}{12} \frac{1}{n^2}
\]

where \( \tilde{M} = \max_{[a,b]} |f''(x)| \).

**Ex:** Compute approximately the integral

\[
I = \int_{2}^{5} \frac{1}{x} \, dx
\]

using left point and trapezoidal rule using \( n+1 \) equally spaced sample points

\[
2 = x_0 < x_1 < \ldots < x_{n+1} = 5
\]

We have \( x_0 = 2 \), \( x_1 = 2 + h \), \( x_2 = 2 + 2h \), \ldots, \( x_n = 2 + nh \).

\[
h = \frac{5 - 2}{n} = \frac{3}{n}
\]
Thus, \( x_k = 2 + k \frac{3}{n} \).

**Left point:**

\[
\int_2^5 \frac{1}{x} \, dx \approx \sum_{k=0}^{n-1} h \frac{f(x_k)}{f(x)} = \frac{3}{n} \sum_{k=0}^{n-1} \frac{1}{x_k}
\]

rectangle over the interval \([x_k, x_{k+1}]\)

\[
= \frac{3}{n} \sum_{k=0}^{n-1} \frac{1}{2 + \frac{3k}{n}}.
\]

**Trapezoid:**

\[
\int_2^5 \frac{1}{x} \, dx \approx \sum_{k=0}^{n-1} h \left( \frac{f(x_k) + f(x_{k+1})}{2} \right)
\]

\[
= \frac{3}{2n} \sum_{k=0}^{n-1} \left( \frac{1}{x_k} + \frac{1}{x_{k+1}} \right).
\]

**How to enter this sum in Matlab?**

Note that Matlab indexes entries of an array by 1,2,3,... not 0,1,2,...

Type \( h = 3/n \)

\( x = 2:h:5 \)

At this time, Matlab understands that \( x(1) = 2, x(2) = 2+h, \ldots \)

We need to adjust \( x_k = 2 + \frac{3(k-1)}{n} \) for \( k = 1,2,\ldots,n+1 \).

See Matlab code on class website.