Some review problems for Final

1. Consider function \( f(z) = \frac{\text{Log}(z+5)}{\sin z} \).

(a) Determine all singular point(s) of \( f \) enclosed in the circle \( C_4(0) \).

(b) Are they isolated singularities? If so, which kind of isolated singularity are they (removable, pole, essential)?

(c) Compute the residue of \( f \) at each of these singularities.

(d) Evaluate the integral \( \int_{\gamma} f(z)dz \) where \( \gamma \) is the circle \( C_4(0) \) oriented counterclockwise.

2. Find the following limits.

(a) \( \lim_{z \to i} \frac{z^4-1}{z-i} \)

Hint: factor or use L’Hospital rule.

(b) \( \lim_{z \to 1+i} \frac{z^2+z-1-3i}{z^2-2z+2} \)

(d) \( \lim_{z \to \infty} \frac{z}{e^z} \)

3. Find an antiderivative if exists of the following functions. Specify the domain of that antiderivative.

(a) \( f(z) = -2(xy + x) + i(x^2 - 2y - y^2) \)

Hint: write \( F' = f \). Then use Cauchy-Riemann equations.

(c) \( f(z) = \bar{z} \)

(e) \( f(z) = \frac{1}{z^2+1} \)

(d) \( f(z) = \frac{z-2}{z^2-z} \)

(b) \( f(z) = z \text{Log } z \)

Hint: first regard \( z \) as real. Do integration by part. Double check formula by differentiation.

(f) \( f(z) = \frac{z+1}{(z-\frac{1}{2})^2 \sin z} \)

(g) \( f(z) = \frac{z+2}{z^2} \)

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4. Compute the following integrals. If you need to use a named theorem, make sure to specify it (Cauchy-Goursat, Fundamental theorem of Calculus, Cauchy’s Integral formula, Cauchy’s Residue theorem.)

(a) \( \int_{\gamma} z^2dz \) where \( \gamma(t) = (\sin t, t^2), \ 0 \leq t \leq \pi. \)

(b) \( \int_{\gamma} \frac{1}{z}dz \) where \( \gamma(t) = e^{3it}, \ 0 \leq t \leq 2\pi. \) (Warning: \( \gamma \) is not a simple loop.)

(c) \( \int_{\gamma} \bar{z}dz \) where \( \gamma(t) = (3t, t^2), \ -1 \leq t \leq 2. \)

(d) \( \int_{\gamma} \frac{e^z}{z(z-3)}dz \) where \( \gamma \) is the unit circle oriented clockwise.

(e) \( \int_{\gamma} z^2 \sin \left( \frac{1}{z^2} \right)dz \)

where \( \gamma \) is the boundary of square with vertices at \( \pm 1 \pm i \) negatively oriented. Hint: use Laurent series.

(f) \( \int_{\gamma} \frac{z+1}{(z-\frac{1}{2})^2 \sin z}dz \) where \( \gamma \) is the circle \( C_2(0) \) oriented counterclockwise.

(g) \( \int_{\gamma} \frac{z+2}{z^2}dz \) where \( \gamma \) is the figure eight curve as in the picture.
Answer key

1. (a) \( z = 0, -\pi, \pi \)
   (b) Yes. Each is a pole of order 1 (single pole).
   (c) \( \text{Res}[f; 0] = \ln 5, \quad \text{Res}[f; \pi] = -\ln(5 + \pi), \quad \text{Res}[f; -\pi] = -\ln(5 - \pi) \)
   (d) \( 2\pi i \ln \frac{5}{25 - \pi^2} \)

2. (a) \(-4i\)
   (b) \(1 - \frac{3}{2}i\)
   (c) 0
   (d) DNE

3. (a) \( F(z) = u + iv \) where \( u(x, y) = -x^2y - x^2 + y^2 + \frac{y^3}{3} \) and \( v(x, y) = \frac{x^3}{3} - 2xy - xy^2 \), valid on \( \mathbb{C} \).
   (b) \( F(z) = -\frac{x^2}{4} + \frac{x^2}{2} \log z \), valid on \( \mathbb{C} \setminus \mathbb{R} \leq 0 \).
   (c) No antiderivatives
   (d) \( F(z) = -\log(z - 1) + 2\log z \), valid on \( \mathbb{C} \setminus \mathbb{R} \leq 1 \).
   (e) \( F(z) = \frac{1}{2i} \log(z - i) - \frac{1}{2i} \log(z + i) \), valid on \( \mathbb{C} \) minus to lines \( \{t + i : t \leq 0\} \) and \( \{t - i : t \leq 0\} \).

4. (a) \(-\frac{\pi^6}{3}i\)
   (b) \(6\pi i\)
   (c) \(21 + 9i\)
   (d) \(\frac{2\pi i}{3}\)
   (e) \(\frac{\pi i}{3}\)
   (f) \(2\pi i(1 + \frac{1}{\pi^2})\)
   (g) \(-4\pi i\)

Formula to be provided on Final Exam

\[
e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots
\]
\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots
\]
\[
\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots
\]
\[
\log(z + 1) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots
\]

Cauchy–Riemann equations:

\[
\begin{cases}
\partial_x u = \partial_y v \\
\partial_y u = -\partial_x v
\end{cases}
\]

Cauchy’s Integral formula:

\[
\int_{\gamma} \frac{f(z)}{(z - a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)
\]
Cauchy’s Residue formula:

\[ \int_{\gamma} f(z) \, dz = 2\pi i (\text{Res} \ [f; z_1] + \text{Res} \ [f; z_2] + \cdots + \text{Res} \ [f; z_m]) \]

If \( a \) is a pole of order \( n \) of function \( f \) then

\[ \text{Res} \ [f; a] = \frac{1}{(n - 1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}}[(z - a)^n f(z)] \]