Homework 3
Due 4/25/2019

Problems 1, 2, 3 are computational. Make sure to write down every computational step. You can double check your answers using Mathematica, for example,

\[ N[\sin(1 - i)] \]
\[ 1.29846 - 0.634964 \, \text{i} \]

1. Write the following complex numbers in standard form. Use the principal branch of the logarithm if necessary.

   (a) \(4^i\)  
   (b) \((1 + i\sqrt{3})^{i/2}\)  
   (c) \((-1)^{\sqrt{2}}\)  
   (d) \((i - 1)^{2i + 3}\)  
   (e) \(e^{\sin i}\)  
   (f) \(\cos(-2 + i)\)  
   (g) \(\sin\left(\frac{\pi + 4i}{4}\right)\)  
   (h) \(\tan\left(\frac{\pi + 2i}{4}\right)\)  
   (i) \(\cosh\left(\frac{1-i\pi}{4}\right)\)  
   (j) \(\sin(1 + i\pi)\)

2. Find all complex values of the following:

   (a) \(\log(\sqrt{3} - i)\)  
   (b) \(\log(-ie)\)  
   (c) \(\arcsin(1 + i)\)

3. Determine if each of the following statements is true. If it is, prove it. If it is not, give a counterexample.

   (a) \(\sqrt{\sqrt{-z}} = i\sqrt{\sqrt{z}}\) (principal branch being used)  
   (b) \(\cos^2 z + \sin^2 z = 1\)  
   (c) \(|\cos z|^2 + |\sin z|^2 = 1\)  
   (d) \(\sin(z + 2\pi) = \sin z\)  
   (e) \(\sin(\pi - z) = \sin z\)  
   (f) \(e^z = \overline{e^z}\)

4. Let \(f(z)\) be a branch of the multi-valued function \(z^{1/4}\) such that \(f(1) = -i\). Find \(f(4i)\).

5. Let \(f(z) = \sqrt[4]{z - 1} \sqrt[4]{z - i}\) where each root function is defined using the principal logarithm (whose domain is \(\mathbb{C} \setminus \mathbb{R}_{\leq 0}\)). Find the domain of \(f\).

6. To each multi-valued function in Part (a) and (b), define a continuous single-valued branch for it using the following procedure:

   - Choose a branch for the logarithm. Recall that this amounts to choosing a branch for the argument function \(\arg z\). Restricting the argument to any interval of length \(2\pi\), for example, \((-\pi, \pi]\), \([0, 2\pi)\), \([\pi/4, 7\pi/4)\), etc would each give a branch.
   - Cut the complex plane: each interval chosen for the argument corresponds to a curve to be removed from the complex plane. For example, if the interval \([0, 2\pi)\) is chosen, the curve to remove is \(\{z : \text{Arg} z = 0\} \cup \{0\}\).
   - Determine the domain of this single-valued function. That is, the set of all \(z\)'s such that the expression inside the logarithm does not lie on the branch cut. (Geometry is useful.)
   - Testing: pick two points in the domain of the above function \(f(z)\) (say, \(z_1 = 2i + 1\) and \(z_2 = -2 + i\) if possible). Compute \(f(z_1)\) and \(f(z_2)\).
(Note that Step 2 and 3 would not be necessary if the continuity requirement is removed.)

(a) \( \log(z^5) \)
(b) \( \log(z^2 + 1) \)

The purpose of the following problem is to use Mathematica to visualize some mapping properties of the function \( f(z) = \sin z \). Make sure to write the Mathematica code you use, give explanation and some comments on the graph. Similar properties of the exponential function \( e^z \) are addressed in the supplemental material “Mapping properties via Mathematica” posted on the course website.

From Problem 2, we know that \( f(z + 2\pi) = f(z) \) and \( f(\pi - z) = f(z) \). To consider some mapping properties of \( f(z) \), we restrict the domain of \( f \) to the infinite vertical strip \( [-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty) \).

7. (a) Draw the image under \( f \) of the rectangle \( [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-1, 1] \).

(b) Draw the images under \( f \) some grid lines (horizontal and vertical). Based on the picture, what are the images of the line \( x = -\frac{\pi}{2} \) and \( x = \frac{\pi}{2} \) ?

(c) What region does \( f \) map the vertical strip \( [-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty) \) onto? Is it also a one-to-one mapping?

(d) What region does \( f \) map the open vertical strip \( (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, \infty) \) onto? Is it also a one-to-one mapping?