Inverse of sine:
\[
\arcsin z = w \implies \sin w = z \implies \frac{e^{iw} - e^{-iw}}{2i} = z
\]

Put \( u = e^{iw} \). Then \( \frac{u - u^{-1}}{2i} = z \implies u^2 - 2uiz + 1 = 0 \implies u = iz + \sqrt{1-z^2}
\]

\[
\implies iw = \log u = \log (iz + \sqrt{1-z^2})
\]

\[
\implies \arcsin z = \frac{1}{i} \log (iz + \sqrt{1-z^2})
\]

Multivalues come from the log and \( \Gamma \) functions.

Similarly,
\[
\arccos z = \frac{1}{i} \log (z + i\sqrt{1-z^2})
\]

\[
\arctan z = \frac{i}{2} \log \left( \frac{i+z}{i-z} \right)
\]

\[
\text{Ex: compute } \arcsin i
\]

\[
\arcsin i = \frac{1}{i} \log \left( i + \sqrt{1-i^2} \right) = \frac{1}{i} \log \left( -1 + \sqrt{2i} \right)
\]

\[
\text{complex root} = \frac{1}{i} \log \left( -1 + \sqrt{2i} \right)
\]

\[
\text{real root} = \frac{1}{i} \log \left( -1 + i \sqrt{2} \right)
\]

If plus sign is selected,
\[
\arcsin i = \frac{1}{i} \log \left( -1 + i \sqrt{2} \right) = \frac{1}{i} \left( (\ln(-1+i\sqrt{2}) + i \cdot 2\pi) \right)
\]

\[
= 2\pi i - (\ln(-1+i\sqrt{2})
\]

If minus sign is selected,
\[
\arcsin i = \frac{1}{i} \log \left( -1 - i \sqrt{2} \right) = \frac{1}{i} \left( (\ln(1+i\sqrt{2}) + c\pi + i \cdot 3\pi) \right)
\]

\[
= -\pi i - (\ln(1+i\sqrt{2})
\]
\[ = k2\pi + \pi - i\ln(1+\sqrt{2}) \]

Conclusion: \[ \arcsin i = \left\{ k2\pi + \pi - i\ln(1+\sqrt{2}) : k \in \mathbb{Z} \right\} \]
[\[ = \left\{ n\pi - i(-1)^n\ln(1+\sqrt{2}) : n \in \mathbb{Z} \right\} \]

* How to define a single branch for the \( \arcsin z \)?

\[ \text{Arcsin } z = \frac{1}{i} \log \left( \frac{z^2 + \sqrt{1-z^2}}{\sqrt{1-|z|^2}} \right) \quad \text{principal branch} \]

| -1 | 1 |

Up to this point, we have considered some basic functions on \( \mathbb{R} \) and their extensions to \( \mathbb{C} \):

\[ \text{exp, sin, cos, tan, power } (z^n, n \in \mathbb{Z}) : \text{ single-valued functions} \]

\[ \text{log, arcsin, arcos, arctan } \quad z^a (a \in \mathbb{C}) : \text{ multi-value functions} \]

Branch cuts are introduced to create a single-valued function (branch) out of a multi-valued function (alike to a tree). There are many ways to define a branch. Think of a sphere: the sphere itself is not a graph of a function. But a hemisphere is. It is also a maximal branch in sense that one can't extend the hemisphere and still have a graph of a function.

The functions have much richer geometric properties than their real counterparts (as already seen in previous lectures and homework). For example, these functions are conformal transformations (i.e. angle-preserving).
To explain these geometric properties analytically, we need more tools, for example, continuity, derivatives, integrals... These concepts are defined on functions $f: \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$.

A subset $\mathbb{D} \subset \mathbb{C}$ has richer geometric properties than a subset $\mathbb{C} \subset \mathbb{R}$. These properties of $\mathbb{D}$ influence all functions defined on it.

We introduce some geometric/topological properties as follows:

- **Circle centered at $z_0$ with radius $r$** : $C_r(z_0)$

  Note that the textbook uses notation $C(z_0, r)$.

  $C_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| = r \}$

  ![Circle](circle.png)

- **Open disk centered at $z_0$ with radius $r$** : $D_r(z_0)$

  The textbook uses notation $D(z_0, r)$.

  $D_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| < r \}$

  ![Open Disk](open_disk.png)

- **Closed disk** : $\overline{D}_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| \leq r \}$

- **Let $G \subset \mathbb{C}$**

  A point $a \in G$ is said to be an interior point of $G$ if $D_r(a) \subset G$ for some $r > 0$.  

  ![Interior Point](interior_point.png)