A function \( f(z) \) can be viewed as a pair \((u(x,y), v(x,y))\)
where
\[
\begin{align*}
u(x,y) &= \text{Re}(f(z)) \\
u(x,y) &= \text{Im}(f(z))
\end{align*}
\]

Observation: \( \lim_{z \to a} f(z) = L = (L_1, L_2) \) if and only if
\[
\begin{align*}
\lim_{(x,y) \to (x_0,y_0)} u(x,y) &= L_1 \\
\lim_{(x,y) \to (x_0,y_0)} v(x,y) &= L_2
\end{align*}
\]
where \( a = (x_0, y_0) \)

In other words, the real part approaches the real part;
the imaginary part approaches the imaginary part.

* Continuity:
\[
f : G \subset \mathbb{C} \to \mathbb{C} \text{ is said to be continuous at } z_0 \text{ if } \lim_{z \to z_0} f(z) = f(z_0)
\]

\( f \) is said to be continuous on \( G \) if it is continuous at every point of \( G \).

Thus:

Write \( f(z) = u(x,y) + iv(x,y) \). Then \( f \) is cont. at \( z_0 = (x_0, y_0) \) if and only if both \( u \) and \( v \) are continuous at \( (x_0, y_0) \).

Ex:

\[
f(z) = z^2 \implies \begin{cases}
u(x,y) = x^2 - y^2 \\
v(x,y) = 2xy
\end{cases}
\]

Because \( u \) and \( v \) are continuous, \( f \) is continuous.

Similarly, the function \( g(z) = z^n \) (\( n \in \mathbb{N} \)) is continuous:
\[
g(z) = (x + iy)^n = (\text{---}) + i(\text{---})
\]
polynomials in \( x \) and \( y \).
Ex. \( f(z) = e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y \)

\[
\cos z = \frac{e^z + e^{-z}}{2} = u(x, y) + iv(x, y)
\]

Ex: \( f(z) = \text{Arg}(z) \)

\( z = -2 \)

\[
\text{discontinuous at } -2
\]

\[
\text{Arg}(-2 + i \varepsilon) \approx \pi \quad \text{discontinuous at } -2
\]

\[
\text{Arg}(-2 - i \varepsilon) \approx -\pi
\]

\[
\text{Arg is undefined at } z = 0.
\]

For \( z = x + iy \):

* if \( z \) is in the right half plane,

\[
\arg z = \arctan \left( \frac{y}{x} \right) \implies \text{continuous}
\]

\[
u(x, y) + i \cdot 0
\]

* if \( z \) is in the upper half plane,

\[
\arg z = \arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) \implies \text{cont.}
\]

* if \( z \) is in the lower half plane,

\[
\arg z = -\arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) \implies \text{cont.}
\]

Ex.

\[
\begin{align*}
 f(z) &= \frac{z}{|z|} \\
u(x, y) &= \frac{x}{\sqrt{x^2 + y^2}} \\
u(x, y) &= \frac{-y}{\sqrt{x^2 + y^2}}
\end{align*}
\]

\( u \) is not continuous at \((0, 0)\). Thus, \( f \) is not cont. at 0.
Properties of limits:

Limit of complex functions satisfies the same algebraic properties as limit of real functions.

- **Additive:**
  \[
  \lim_{z \to a} (f(z) + g(z)) = \lim_{z \to a} f(z) + \lim_{z \to a} g(z)
  \]

- **Scalar-multiplicative:**
  \[
  \lim_{z \to a} cf(z) = c \lim_{z \to a} f(z)
  \]

- **Multiplicative:**
  \[
  \lim_{z \to a} f(z) g(z) = \lim_{z \to a} f(z) \cdot \lim_{z \to a} g(z)
  \]

- **Inverse:**
  \[
  \lim_{z \to a} \frac{1}{f(z)} = \frac{1}{\lim_{z \to a} f(z)}
  \]

Parallel properties of continuity:

- \( f, g : \mathbb{C} \to \mathbb{C} \) continuous at \( a \in \mathbb{C} \)
- \( f + g, cf, fg, \frac{f}{g} \) are continuous at \( a \)
- \( f(g(z)) \) is continuous at \( a \) if \( g \) is continuous at \( a \) and \( f \) is continuous at \( f(a) \)

Consequences:

- Rational functions \( \frac{P(z)}{Q(z)} \) are continuous.
- \( \log z = \ln |z| + i \arg z \) is continuous on \( \mathbb{C} \setminus \{0\} \).
- \( z^c = e^{c \log z} \) is continuous on \( \mathbb{C} \setminus \{0\} \).
Consider functions $\gamma : I \subseteq \mathbb{R} \rightarrow \mathbb{C}$. These are curves (paths) on the complex plane.

Circle centered at $z_0$ with radius $r$:

$$\gamma(t) = z_0 + re^{it}, \quad 0 \leq t \leq 2\pi$$

Line passing through $z_0$ with slope $t$:

$$\gamma(t) = z_0 + te^{it}, \quad t \in \mathbb{R}$$

Segment from $z_1$ to $z_2$:

$$\gamma(t) = z_1 + t(z_2 - z_1), \quad 0 \leq t \leq 1$$

Line from $z_1$ to $z_2$: Same formula, but $t \in \mathbb{R}$. 