A function $\gamma : \mathbb{C} \rightarrow \mathbb{C}$ is said to be a path if it is continuous.

How to go travel more quickly (from example going back to the original point after time $\pi$)?

Scale time:

$\gamma(t) = z_0 + re^{it}$ \quad ($0 \leq t \leq \pi$)

How to reverse the direction of $\gamma$?

$\gamma_{\text{rev}}(t) = \gamma(a+b-t)$ \quad ($a \leq t \leq b$)

$\gamma(t) = z_0 + te^{it}$ \quad ($t \in \mathbb{R}$)

$\gamma(t) = z_1 + t(z_2 - z_1)$, \quad $0 \leq t \leq 1$

Ex:

Find the region of continuity of $f(z) = \left( \frac{z}{z+1} \right)^{\frac{1}{2}}$

where the branch $\left[ \frac{\pi}{2}, \frac{5\pi}{2} \right]$ of the argument is used.

By definition,

$f(z) = \exp \left( z \log \frac{z}{z+1} \right)$
where \( \text{Log} \frac{z}{z+1} = \ln |\frac{z}{z+1}| + i \text{arg} \left( \frac{z}{z+1} \right). \)

The rational function \( \frac{z}{z+1} \) is continuous everywhere except at -1.

In addition, need to exclude of \( z \)'s such that \( \frac{z}{z+1} \in i \mathbb{R}^<0. \)

Write: \( \frac{z}{z+1} = it \) where \( t \geq 0 \)

\[ \Rightarrow z = itz + it \]

\[ \Rightarrow z = \frac{it}{1-it} \]

Region of continuity of \( f \) is \( C \) minus the set

\[ \{ z \in \mathbb{C} : z \neq 0 \} \]

How to visualize this set?

Write in standard form:

\[ \frac{it}{1-it} = \frac{it(1+it)}{1+t^2} = -\frac{t^2}{1+t^2} + \frac{t}{1+t^2} \]

\[ \text{this is a curve} \]

On Mathematica:

```mathematica
ParametricPlot[{\frac{-t^2}{1+t^2}, \frac{t}{1+t^2}}, {t, 0, 10}]
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In fact, this is the upper half of the ellipse

\[ (z+\frac{1}{2})^2 + 4y^2 = 1. \]
* Velocity of a path

\[ \gamma : [a,b] \to \mathbb{C} \]

How to define \( \gamma'(t) \)?

Write \( \gamma(t) = x(t) + iy(t) = (x(t), y(t)) \).

\[
\gamma'(t) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{x(t) - x(t_0)}{t - t_0} + i \lim_{t \to t_0} \frac{y(t) - y(t_0)}{t - t_0}
\]

\[ = x'(t_0) + iy'(t_0) \]

\[ \Rightarrow \gamma'(t) = (x'(t), y'(t)) \]

Derivative of a function \( \gamma : I \subset \mathbb{R} \to \mathbb{C} \) is obtained by taking derivative of each component.

Note: Rolle's theorem doesn't hold for complex-valued functions:

\( \gamma(a) = \gamma(b) \) doesn't necessarily imply that there exists \( c \in (a,b) \) such that \( \gamma'(c) = 0 \).

Ex: the circle \( \gamma(t) = e^{it}, 0 \leq t < 2\pi \)

\[
\gamma(0) = \gamma(2\pi) = 1 + 0i
\]

but \( \gamma'(t) = (\cos t)' + i(\sin t)' = -\sin t + i\cos t \neq 0 \) for all \( t \).

A path \( \gamma : [a,b] \to \mathbb{C} \) is said to be smooth if \( \gamma'(t) \) exists and is continuous on \([a,b]\), with the convention that

\[
\gamma'(a) = \lim_{t \to a^+} \frac{\gamma(t) - \gamma(a)}{t - a}
\]

\[
\gamma'(b) = \lim_{t \to b^-} \frac{\gamma(t) - \gamma(b)}{t - b}
\]

\[ \text{smooth} \quad \nRightarrow \quad \text{non-smooth} \]
A path \( \gamma : [a, b] \) is said to be regular if it is smooth and \( \gamma'(t) \neq 0 \) for all \( t \in [a, b] \).

In other words, a regular path never slows down to a stop.

\[ \gamma(t) = (e^t, e^{2t}) \], \(-1 \leq t \leq 1\)

not regular since \( \gamma'(0) = 0 \).

This path slows down to a stop at \( t = 0 \), then it reverses itself.