We considered examples of complex functions being differentiable only on a curve and nowhere else. This region of differentiability is too small to do calculus. A function holomorphic at some point (differentiable on at least a disk) is a more preferred object to study calculus on.

So far, we know 2 methods to check differentiability and compute derivatives: using definition or C-R equations.

Ex: \( f(z) = z^2 \)

1\(^{st}\) method: use definition

\[
\frac{f(z)-f(z_0)}{z-z_0} = \frac{(z-z_0)(z+z_0)}{z-z_0} = z + z_0 \quad \text{as } z \to z_0
\]

Thus, \((z^2)' = 2z\).

2\(^{nd}\) method: use C-R's theorem

\[
z^2 = \frac{x^2-y^2}{u} + i \frac{2xy}{v}
\]

\[
\begin{align*}
\frac{\partial_x u}{\partial_y v} - \frac{\partial_y u}{\partial_x v} &= 2x \\
\frac{\partial_x v}{\partial_y v} - \frac{\partial_y v}{\partial_x v} &= 2y
\end{align*}
\]

C-R eqns. are satisfied

\((z^2)' = \partial_x u + i \partial_y v = 2x + i2y = 2z\).

There are cleaner ways to check differentiability and compute derivatives.

Laws of derivatives

Sum: \((f(z) + g(z))' = f'(z) + g'(z)\)

Product (Leibniz's rule): \((f(z)g(z))' = f(z)g'(z) + f'(z)g(z)\)

Quotient: \((\frac{f(z)}{g(z)})' = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}\)

Composition (chain rule): \([f(g(z))]' = f'(g(z))g'(z)\)

Ex: \((z^n)' = nz^{n-1}\), where \(n = 1, 2, 3, \ldots\) (use product rule)
Example:
\[(\frac{1}{z})' = -\frac{1}{z^2}, \quad \forall z \in \mathbb{C} \setminus \{0\}\]

Use quotient rule.

Example:
\[\log z \quad \rightarrow \quad e^z \quad \rightarrow \quad (\infty, 0) \times (-\pi, \pi) \rightarrow \mathbb{C} \setminus \mathbb{R}_{\leq 0}\]

\[e^{\log z} = z\]

Take derivative of both sides:
\[(\log z)' e^{\frac{\log z}{z}} = 1\]

Principal logarithm is holomorphic on \(\mathbb{C} \setminus \mathbb{R}_{\leq 0}\) and \((\log z)' = \frac{1}{z}\).

Example:
\[f(z) = \sqrt{z} \quad \text{(principal branch)}\]

By definition, \(f(z) = e^{\frac{1}{2} \log z}\).

\(f\) is holomorphic on \(\mathbb{C} \setminus \mathbb{R}_{\leq 0}\) and
\[f'(z) = \frac{1}{2} (\log z)' e^{\frac{1}{2} \log z} = \frac{1}{2z} \frac{\log z}{z} = \frac{1}{2z} \cdot \frac{1}{2z} = \frac{1}{2z} \cdot \frac{1}{2z}\]

Observations:

* Derivative of logarithm is the same \((\frac{1}{z})\), regardless of the chosen branch. (Each branch differs from one another by a constant \(k2\pi i\), whose derivative is zero).

* \((e^z)' = e^z\)

branch of logarithm
Constant functions

Suppose \( f: \mathbb{C} \rightarrow \mathbb{C} \) satisfies \( f'(z) = 0 \) for all \( z \in \mathbb{C} \). Is it possible to conclude that \( f = \text{const} \) on \( \mathbb{C} \)?

This is a basic question to ask before one considers antiderivatives.

\[
\begin{align*}
\int g' &= f' \\
\int h' &= f'
\end{align*}
\]

\( g \) and \( h \) differ from each other by a constant.

Recall how we answer this question in the case \( f: [a, b] \rightarrow \mathbb{R} \).

To show \( f(a) = f(b) \) for any \( a, b \in [a, b] \), we used Lagrange's theorem: if \( f \) is cont. on \([a, b]\) and differentiable on \((a, b)\) then there exists \( c \in (a, b) \) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]

\( \Rightarrow f(b) = f(a) \)

An important ingredient in the above argument is that \( f \) is differentiable on the entire interval \((a, b)\). If \( I \) is not an interval, say \( I = (0, 1) \cup (2, 3) \), then \( f \) wouldn't necessarily be constant on \( I \). It is constant on \((0, 1)\) and on \((2, 3)\) separately.

Connected components of \( I \)
Def: An open set $G \subset \mathbb{C}$ is said to be connected if any two points in $G$ can be connected to each other by a path in $G$.

*Note: this path can be chosen to be a "rectangle" path:

Thm: Let $G \subset \mathbb{C}$ be an open connected subset. A function $f: G \to \mathbb{C}$ with $f'(z) = 0$ for all $z \in G$ must be constant.

why? Recall that $f'(z) = \partial_x u + i \partial_y v = \partial_y v - i \partial_x u$

Thus, $\partial_x u = \partial_y v = \partial_y v = \partial_x u = 0$ everywhere in $G$.

For $z_1, z_2 \in G$, there is a rectangle path in $G$ that connects them. The value of $f$ is the same on each straight segment of the path. Thus, $f(z_2) = f(z_1)$.

Examples of non-connected sets:

$G_1 \cup G_2 \quad f(z) = \begin{cases} \frac{z}{2} & \text{on } G_1 \\ b & \text{on } G_2 \end{cases}$

$G = G_1 \cup G_2 \quad f'(z) = 0 \quad \forall z \in G$

but $f$ is not constant on $G$. 