If $F' = f$ on a connected set $G \subset \mathbb{C}$ then all antiderivatives of $f$ on $G$ is $F(z) + C$.

Why? Suppose $F$ is an antiderivative of $f$ on $G$. Then

$F' = f$ on $G$ implies $(F - F_0)' = 0$ on $G$

$\Rightarrow F - F_0 = \text{const}$ on $G$ since $G$ is connected.

**Ex:** $f(z) = \sqrt{1-z} + \sqrt{1+z}$ (principal branch)

Domain of continuity / differentiability is $\mathbb{C} \setminus \{ z \in \mathbb{R} : z \leq -1 \}$

\[ F(z) = \frac{2}{3} (1-z)^{\frac{3}{2}} + \frac{2}{3} (1+z)^{\frac{3}{2}} \]

This region is connected. Thus, all antiderivatives of $f$ are

$F(z) + C = \frac{2}{3} (1-z)^{\frac{3}{2}} + \frac{2}{3} (1+z)^{\frac{3}{2}} + C$

where $C$ is a complex constant.

**Thm:** (to be proved next week) if $f$ is holomorphic on $G$ and $G$ is simply connected then it has an antiderivative on $G$. 


A simply-connected set is a connected set in which every closed path (loop) can be continuously contracted to a point. Intuitively, a simply-connected set is a connected set without "holes.

Simply-connected

not simply-connected

\( C \setminus \{0\} \)

\( \text{not simply-connected} \)

\( \text{green curve can't contract to a point} \)

\( \text{E.g. the function } \frac{1}{z} \text{ has no antiderivatives on } C \setminus \{0\}, \) although it is continuous on \( C \setminus \{0\} \). Why?

We know that \( \text{Log} z \) is an antiderivative of \( \frac{1}{z} \) on \( C \setminus \{0\} \), but it is not simply connected. An antiderivative is not guaranteed to exist.

\( C \setminus \{0\} \) is not simply connected. An antiderivative is not guaranteed to exist. In this example, it in fact doesn't exist.

\( C \setminus \{0\} \) is simply connected. An antiderivative is guaranteed to exist.

\( \text{jump of } -2\pi i \)

Note that \( C \setminus \{0\} \) is a connected set. Any antiderivative of \( \frac{1}{z} \) on \( C \setminus \{0\} \), if exists, must be equal to \( \text{Log} z + C \) on \( C \setminus \{0\} \). The function \( \text{Log} z + C \) has a jump of \( 2\pi i \) across the ray \( R \setminus \{0\} \). There is no way to "fix" the function \( \text{Log} z + C \) on the ray \( R \setminus \{0\} \) to make it continuous on there (not to say differentiable).

Mapping properties of holomorphic functions.

* Multiplication by complex number: \( z \mapsto az \)
Write \( a = re^{i\theta} \).

The multiplication by complex number \( a = re^{i\theta} \) is obtained by stretching by factor \( r \) and then rotating by angle \( \theta \). The order of rotation and dilation can be reversed.

In complex standard form, \( \wp(z) = (r \cos \theta + ir \sin \theta)(x + iy) = \ldots \)

\( \wp \) is a linear map. In matrix form,

\[
\wp \begin{bmatrix} x \\ y \end{bmatrix} = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

dilation \hspace{1cm} rotation

**Ex:** Rotation by \( \pi/2 \) about the origin followed by translation by vector \((2, 1)\):

\[
\begin{align*}
z & \rightarrow e^{i\pi/2} z \\
2 + i + e^{i\pi/2} \bar{z} & \rightarrow 2 + i + e^{i\pi/2} \bar{z}
\end{align*}
\]
\[\wp(z) = 2 + i + e^{i\pi/2} \bar{z}\]

In standard form:
\[\wp(z) = 2 + i + i(x + iy) = 2 - y + ((1+x))\]

which can be viewed as \( \wp(x, y) = (2 - y, 1 + x) \).

* General holomorphic map \( f: \mathbb{C} \rightarrow \mathbb{C} \):
\[ f'(z_0) = \frac{af'(z_0)}{dz_0} \]

\[ af(z_0) = f'(z_0) dz_0 \]

(In precise form, \( df = f'(z_0) dz \).)

Write \( f'(z_0) = re^{i\theta} \)

\( df \) is obtained by dilation \( dz \) by factor \( r \) and rotation by angle \( \theta \).

Let \( \gamma \) be a curve passing through \( z_0 \). Suppose \( \gamma(0) = z_0 \).

The image of \( \gamma \) under \( f \) is another curve: \( \eta(t) = f(\gamma(t)) \).

\[ \eta'(0) = \frac{f'(\gamma(0)) \gamma'(0)}{\text{tan} \text{gent vector of } \gamma \text{ at } z_0 \text{. }} \]

\[ \eta' \text{ at } f(z_0) = re^{i\theta} \]

\( \eta'(0) \) is obtained from \( \gamma'(0) \) by scaling with factor \( r \) and rotating by angle \( \theta \).

The angle between \( \eta_1'(0) \) and \( \eta_2'(0) \) is equal to the angle between \( \gamma_1'(0) \) and \( \gamma_2'(0) \).

Def: A function \( f: \mathbb{C} \to \mathbb{C} \) is said to be conformal if \( f \) preserves angles (both size and sign).

Thm: \( f: \mathbb{C} \to \mathbb{C} \) is conformal if and only if

1. \( f \) is differentiable on \( G \) (in other words, holomorphic on \( G \))
2. \( f'(z) \neq 0 \quad \forall z \in G \).